Elements of Convex Analysis and Optimization

Master II Datascience

Pascal Bianchi, Olivier Fercoq, Walid Hachem

November 11, 2020

Contents

Chapter 1

Elements of convex analysis

Let $\mathcal{X}$ be a Euclidian space endowed with a scalar product denoted by $\langle \cdot, \cdot \rangle$ and an associated norm $\| \cdot \|$.

1.1 Convex sets

1.1.1 Separation results

In this section, C denotes a subset of $\mathcal{X}$.

Definition 1.1 (Convex set). The set C is **convex** if

$$\forall (x,y) \in C^2, \forall t \in [0,1], \quad t\,x + (1-t)\,y \in C.$$

Proposition 1.1. *1. Any arbitrary intersection of convex sets is convex.*

2. The closure and the interior of a convex set are convex.

Proof. The proof is left as an exercise. □

Lemma 1.2. *Let C be a convex set. For every $k \geq 1$, every $(a_1, \dots, a_k) \in C^k$ and every $(\alpha_1, \dots, \alpha_k) \in \mathbb{R}_+^k$ such that $\sum_{i=1}^k \alpha_i = 1$,*

$$\sum_{i=1}^k \alpha_i\, a_i \;\in\; C\;.$$

Such a weighted sum, with non negative weights $\alpha_1, \dots, \alpha_k$ summing to one, is called a **convex combination** of the points $a_1, \dots, a_k$.

Proof. By induction. The statement holds for $k = 1$. Now consider $k \geq 2$ and assume that $a_k > 0$.

$$\sum_{i=1}^k \alpha_i\, a_i = \alpha_k\, a_k + (1-\alpha_k) \sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k}\, a_i\,.$$

The sum in the righthand side lies in C by induction. Thus, the righthand side lies in C by definition of a convex set. □

Proposition 1.3 (Projection). *Let $C \subset \mathcal{X}$ be a non-empty, closed, and convex set. For every $x \in \mathcal{X}$, there is a unique point in C, denoted by $P_C(x)$, such that*

$$\text{for all } y \in C, \ \|y - x\| \geq \|P_C(x) - x\|.$$

Moreover, the mapping $P_C : \mathcal{X} \to \mathcal{X}$ satisfies the following.

1. $\forall y \in C$, $\langle y - P_C(x), x - P_C(x)\rangle \le 0$.

2. $\forall (x,y) \in \mathcal{X}^2$, $\|P_C(y) - P_C(x)\| \le \|y - x\|$.

The point $P_C(x)$ is called the projection of x on C.

Proof.
1. Let $d_C(x) = \inf_{y\in C} \|y - x\|$. There exists a sequence $(y_n)_n$ in C such that $\|y_n - x\| \to d_C(x)$. The sequence is bounded, so extract a subsequence which converges to y_0. By continuity of $y \mapsto \|y - x\|$, we have $\|y_0 - x\| = d_C(x)$, as required.
To prove uniqueness, consider a point $z \in C$ such that $\|z - x\| = d_C(x)$. By convexity of C, $w = (y_0 + z)/2 \in C$, so $\|w - x\| \ge d_C(x)$. According to the parallelogram identity [1],

$$\begin{aligned} 4d_C(x)^2 &= 2\|y_0 - x\|^2 + 2\|z - x\|^2 \\ &= \|y_0 + z - 2x\|^2 + \|y_0 - z\|^2 \\ &= 4\|w - x\|^2 + \|y_0 - z\|^2 \\ &\ge 4d_C(x) + \|y_0 - z\|^2. \end{aligned}$$

Thus, $\|y_0 - z\| = 0$ and $y_0 = z$.

2. Let $p = P_C(x)$ and let $y \in C$. For $\epsilon \in [0,1]$, let $z_\epsilon = p + \epsilon(y - p)$. By convexity, $z_\epsilon \in C$. Consider the function 'squared distance from x':

$$\varphi(\epsilon) = \|z_\epsilon - x\|^2 \;=\; \|\epsilon(y - p) + p - x\|^2.$$

For $0 < \epsilon \le 1$, $\varphi(\epsilon) \ge d_C(x)^2 = \varphi(0)$. Furthermore, for ϵ sufficiently close to zero,

$$\varphi(\epsilon) = d_C(x)^2 - 2\epsilon \langle y - p, x - p\rangle + o(\epsilon),$$

whence $\varphi'(0) = -2\langle y - p, x - p\rangle$. In the case $\varphi'(0) < 0$, we would have, for ϵ close to 0, $\varphi(\epsilon) < \varphi(0) = d_C(x)$, which is impossible. So $\varphi'(0) \ge 0$ and the result follows.

3. Adding the inequalities

$$\begin{aligned} &\langle P_C(y) - P_C(x), x - P_C(x)\rangle \le 0\,, \text{ and} \\ &\langle P_C(x) - P_C(y), y - P_C(y)\rangle \le 0\,, \end{aligned}$$

yields $\langle P_C(y) - P_C(x), y - x\rangle \ge \|P_C(x) - P_C(y)\|^2$. The conclusion follows using the Cauchy-Schwarz inequality. □

Proposition 1.4 (Separation of a point and a convex set by an hyperplane)**.** *Let C by a non empty, closed and convex set. Consider $x_0 \in \mathcal{X}\backslash C$. Then, there exists $a \in \mathbb{R}$ and $w \in \mathcal{X}\backslash\{0\}$ such that*

$$\begin{aligned} \forall x \in C,\ \langle w, x\rangle + a &\le 0\,, \\ \langle w, x_0\rangle + a &> 0\,. \end{aligned}$$

Proof. Set $w = x_0 - P_C(x_0)$ and $a = \langle w, P_C(x_0)\rangle$. Note that $w \ne 0$ and $\langle w, x_0\rangle + a = \|w\|^2 > 0$. For every $x \in C$, $\langle w, x - x_0\rangle \le 0$ by Proposition 1.3. □

[1] $2\|a\|^2 + 2\|b\|^2 = \|a + b\|^2 + \|a - b\|^2$.

The set $H := \{x \in \mathcal{X} : \langle w, x\rangle + a = 0\}$ defines an hyperplane. This hyperplane splits the space $\mathcal{X}$ into two half spaces. Proposition 1.4 states that C is included in one of these half spaces, while the point x_0 lies in the (interior of) the other. Otherwise stated, x_0 is *separated* from C by the hyperplane H.

We denote by $\mathrm{cl}(C)$ the closure of a set C and by $\mathrm{int}(C)$ its interior. We denote by $\mathrm{bdry}(C)$ the boundary of a set C, defined by $\mathrm{bdry}(C) = \mathrm{cl}(C) \setminus \mathrm{int}(C)$.

Theorem 1.5 (Supporting hyperplane)**.** *Let C be a non-empty convex set and let $x_0 \in \mathrm{bdry}(C)$. There exists $w \in \mathcal{X} \setminus \{0\}$ such that $\forall x \in C,\ \langle w, x - x_0\rangle \le 0$.*

Proof. Let C and x_0 as in the statement. There is a sequence (x_n) with $x_n \in \mathcal{X} \setminus \mathrm{cl}(C)$ and $x_n \to x_0$, otherwise there would be a ball included in C that would contain x_0, and x_0 would be in $\mathrm{int}(C)$. By Prop. 1.4,

$$\forall n,\ \exists w_n \in \mathcal{X} \setminus \{0\}\ ,\ \forall x \in C, \langle w_n, x\rangle < \langle w_n, x_n\rangle\ . \tag{1.1.1}$$

It can be assumed without restriction that $\|w_n\| = 1$ (otherwise, just replace w_n by $w_n/\|w_n\|$) in the above statement. Since the sequence (w_n) is bounded, we can extract a subsequence $(w_{k_n})_n$ that converges to some $w \in \mathcal{X}$. By continuity of the norm, $\|w\| = 1$. Letting $n \to \infty$ in (1.1.1), we obtain the result. □

1.1.2 Relative interior

Definition 1.2. A set $E \subset \mathcal{X}$ is called an **affine space** if, for all $(x, y) \in E^2$ and for all $t \in \mathbb{R}$, $x + t(y - x) \in E$.

If E is a set and $x \in \mathcal{X}$ is a point, then we define the sum $a + E$ as the set of points of the form $a + x$ for $x \in E$. It is easy to check that if E is an affine space and if $x_0 \in E$, then $E - x_0$ (that is, the set of points of the form $x - x_0$ where $x \in E$) is a vector space, which does not depend on the choice of x_0 in E. The **dimension** of the affine space is the dimension of the corresponding vector space.

Definition 1.3. The **affine hull** $\mathrm{aff}(C)$ of a set $C \subset \mathcal{X}$ is the smallest affine space that contains C.

Definition 1.4. Let $C \subset \mathcal{X}$. The **relative interior** of C, denoted by $\mathrm{ri}(C)$ is the set of points $x \in C$ which admit a neighborhood V such that $V \cap \mathrm{aff}(C) \subset C$.

n.b.: For the students aware of topological notions, $\mathrm{ri}(C)$ is the interior of C in the topology induced by $\mathrm{aff}(C)$. Obviously, $\mathrm{int}(C) \subset \mathrm{ri}(C)$.

Theorem 1.6. *Let C be a non empty and convex set of $\mathcal{X}$, then $\mathrm{ri}(C) \neq \emptyset$.*

Proof. It is not difficult to prove that for every $x_0 \in \mathcal{X}$, $\mathrm{ri}(x_0 + C) = x_0 + \mathrm{ri}(C)$. Hence, it is sufficient to make the proof in the case where $0 \in C$. In this case, $\mathrm{aff}(C)$ is a vector space. Denote by d its dimension. One can construct an independent family $(z_i)_{1 \le i \le d}$ of elements of C (construct it by yourself, as an exercise). Let $\bar{x} = \frac{1}{d+1}\sum_{i=1}^d z_i$ be the barycenter of the points, $z_1, \dots, z_d$ and 0. As $0 \in C$, the point $\bar{x}$ is a convex combination of elements of C, it thus lies in C by Lemma 1.2. Let $\varepsilon > 0$ and consider the neigborhood V of $\bar{x}$ defined as the open ball of radius ε and centered at $\bar{x}$. Choose $x \in V \cap \mathrm{aff}(C)$. As $x \in \mathrm{aff}(C)$, the point x writes as a linear combination of the vectors $(z_i)_i$, say $x = \sum_{i=1}^d w_i z_i$ for some $(w_1, \dots, w_d) \in \mathbb{R}^d$. For every $i = 1, \dots, d$,

$$|w_i - (d+1)^{-1}|\,\|z_i\| \le \|x - \bar{x}\| < \varepsilon\,.$$

Choose $\varepsilon < \min_i \|z_i\|(d(d+1))^{-1}$. Then, for every i,

$$\frac{1-1/d}{d+1} < w_i < \frac{1+1/d}{d+1}.$$

Thus, $w_i > 0$ and $\sum_i w_i < 1$. By Lemma 1.2 again, $x \in C$. This shows that $V \cap \text{aff}(C) \subset C$. Hence, $\bar{x} \in \text{ri}(C)$. $\square$

Theorem 1.7. *Let $M : \mathcal{X} \to \mathcal{Y}$ be a linear operator and C be a convex subset of $\mathcal{X}$. Then,* $\text{ri}(MC) = M\,\text{ri}(C)$.

Proof. To be completed. See (Rockafellar, 2015, Th. 6.6). $\square$

1.2 Convex functions

1.2.1 Definition and properties

For all $f : \mathcal{X} \to [-\infty, +\infty]$, the **domain** of f, denoted by $\text{dom}(f)$, is the set of points x such that $f(x) < +\infty$.
A function f is called **proper** if $\text{dom}(f) \neq \emptyset$ (i.e $f \not\equiv +\infty$) and if f *never* takes the value $-\infty$.

Definition 1.5. Let $f : \mathcal{X} \to [-\infty, +\infty]$. The **epigraph of** f, denoted by $\text{epi}\, f$, is the subset of $\mathcal{X} \times \mathbb{R}$ defined by:

$$\text{epi}\, f = \{(x,t) \in \mathcal{X} \times \mathbb{R} : \quad t \geq f(x)\ \}.$$

Definition 1.6 (Convex function)**.** $f : \mathcal{X} \to [-\infty, +\infty]$ is **convex** if its epigraph is convex.

Proposition 1.8. *A function $f : \mathcal{X} \to [-\infty, +\infty]$ is convex if and only if*

$$\forall (x,y) \in \text{dom}(f)^2,\ \forall t \in (0,1), \quad f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Proof. Assume that f satisfies the inequality. Let (x,u) and (y,v) be two points of the epigraph : $u \geq f(x)$ and $v \geq f(y)$. In particular, $(x,y) \in \text{dom}(f)^2$. Let $t \in]0,1[$. The inequality implies that $f(tx + (1-t)y) \leq tu + (1-t)v$. Thus, $t(x,u) + (1-t)(y,v) \in \text{epi}(f)$, which proves that $\text{epi}(f)$ is convex.
Conversely, assume that $\text{epi}(f)$ is convex. Let $(x,y) \in \text{dom}(f)^2$. For (x,u) and (y,v) two points in $\text{epi}(f)$, and $t \in [0,1]$, the point $t(x,u) + (1-t)(y,v)$ belongs to $\text{epi}(f)$. So, $f(t(x + (1-t)y) \leq tu + (1-t)v$. If $f(x)$ et $f(y)$ are $> -\infty$, we can choose $u = f(x)$ and $v = f(y)$, which demonstrates the inequality. If $f(x) = -\infty$, we can choose u arbitrary close to $-\infty$. Letting u go to $-\infty$, we obtain $f(t(x + (1-t)y) = -\infty$, which proves the statement. $\square$

Lemma 1.9. *If $f : \mathcal{X} \to [-\infty, +\infty]$ is convex, then* $\text{dom}(f)$ *is convex.*

Proof. To do as an exercise. $\square$

Proposition 1.10. *Let $f : \mathcal{X} \to (-\infty, +\infty]$ be a convex function. Then, f is continuous at every point in* $\text{int}(\text{dom}\, f)$.

Proof. To do as an exercise. $\square$

1.2.2 Operations preserving convexity

In the sequel, we use the convention that the supremum of the empty set in $\mathbb{R}$ is equal to $-\infty$.

Definition 1.7. The **upper hull** of a collection of functions $(f_\alpha : \alpha \in I)$ on $\mathcal{X} \to [-\infty, +\infty]$, where I is an arbitrary set, is the function $x \mapsto \sup_{\alpha \in I} f_\alpha(x)$.

Proposition 1.11. *The upper hull of of familly of l.s.c. functions is l.s.c.*
The upper hull of of familly of convex functions is convex.

Proof. Let f denote the upper hull of a collection $(f_\alpha : \alpha \in I)$. We remark that $\operatorname{epi}(f) = \bigcap_\alpha \operatorname{epi}(f_\alpha)$. If every function f_α is l.s.c (resp. convex), then $\operatorname{epi}(f_\alpha)$ is closed (resp. convex). Now $\operatorname{epi}(f)$ is closed (resp. convex) as an arbitrary intersection of closed (resp. convex) sets. Thus, f is l.s.c. (resp. convex). □

Proposition 1.12. *Let $F : \mathcal{X} \times \mathcal{Y} \to [-\infty, \infty]$ be a convex function. Then, the function defined on $\mathcal{X} \to [-\infty, +\infty]$ by $y \mapsto \inf_{x \in \mathcal{X}} F(x, y)$ is convex.*

Proof. Consider $y_1, y_2 \in \mathcal{X}$. Set $\varepsilon > 0$. By definition of the infimum, there exists x_1 (resp. x_2) such that $F(x_1, y_1) \le f(y_1) + \varepsilon$ (resp. $F(x_2, y_2) \le f(y_2) + \varepsilon$). Therefore,

$$\begin{aligned}
f(ty_1 + (1-t)y_2) &= \inf_{x \in \mathcal{X}} F(x, ty_1 + (1-t)y_2) \\
&\le F(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \qquad \text{(by definition if the infimum)} \\
&\le tF(x_1, y_1) + (1-t)F(x_2, y_2) \qquad \text{(by convexity)} \\
&\le tf(y_1) + (1-t)f(y_2) + \varepsilon\,.
\end{aligned}$$

We have shown that for all $\varepsilon > 0$, $f(ty_1 + (1-t)y_2) \le tf(y_1) + (1-t)f(y_2) + \varepsilon$. Letting ε tend to zero, it follows that $f(ty_1 + (1-t)y_2) \le tf(y_1) + (1-t)f(y_2)$. □

Definition 1.8. A map $A : \mathcal{X} \to \mathcal{Y}$ is said **affine** if there exists a linear operator $M : \mathcal{X} \to \mathcal{Y}$ and a vector $b \in \mathcal{Y}$ such that $A : x \mapsto Mx + b$.

Proposition 1.13. *Let $f : \mathcal{Y} \to [-\infty, +\infty]$ be a convex function, and let $A : \mathcal{X} \to \mathcal{Y}$ be an affine map. Then, $f \circ A$ is convex.*

Proof. Let $(x, y) \in \mathcal{X}^2$ and $t \in (0, 1)$. By Prop. 1.8, $f \circ A(tx + (1-t)y) = f(tA(x) + (1-t)A(y)) \le tf(A(x)) + (1-t)f(A(y))$, which proves that $f \circ A$ is convex. □

Proposition 1.14. *Let $m \ge 1$ be an integer and let $f_1, \cdots, f_m$ be convex functions on $\mathcal{X} \to (-\infty, +\infty]$. Then, $\sum_i f_i$ is convex.*

Proof. The point is easily shown from Prop. 1.8. □

Definition 1.9. The infimal convolution of two functions $f, g : \mathcal{X} \to [-\infty, +\infty]$ is the function $f \,\square\, g : \mathcal{X} \to [-\infty, +\infty]$ defined by

$$f \,\square\, g \ :\ y \mapsto \inf_{x \in \mathcal{X}} f(x) + g(y - x)\,. \tag{1.2.1}$$

We say that $f \,\square\, g$ is *exact in a point* y if the infimum in (1.2.1) is reached. We say that $f \,\square\, g$ is *exact* if it is exact in every point. It is clear that $f \,\square\, g = g \,\square\, f$.

Proposition 1.15. *Consider two convex functions $f, g : \mathcal{X} \to (-\infty, +\infty]$. Then, $f \,\square\, g$ is convex.*

Proof. By Prop. 1.13 and Prop. 1.14, the map F defined on $\mathcal{X} \times \mathcal{X}$ by $F : (x, y) \mapsto f(x) + g(y - x)$ is convex. Hence, $f \,\square\, g$ is convex by Prop. 1.12. □

Definition 1.10. The **infimal postcomposition** of a linear operator $M : \mathcal{X} \to \mathcal{Y}$ and a function $f : \mathcal{X} \to [-\infty, +\infty]$ is the function $M \triangleright f : \mathcal{Y} \to [-\infty, +\infty]$ defined by

$$M \triangleright f : y \mapsto \inf\{f(x) \,:\, x \in \mathcal{X},\, Mx = y\}\,.$$

Proposition 1.16. *Consider $f : \mathcal{X} \to (-\infty, +\infty]$ and a linear operator $M : \mathcal{X} \to \mathcal{Y}$. Then, $\operatorname{dom}(M \triangleright f) = M \operatorname{dom} f$. Moreover, if f is convex, then $M \triangleright f$ is convex.*

Proof. The first point follows directly from the definition. Assume that f is convex. Then, the mapping F defined on $\mathcal{X} \times \mathcal{Y}$ by $F : (x, y) \mapsto f(x) + \iota_{\operatorname{gra}(M)}(x, y)$ is convex, where $\operatorname{gra}(M) = \{(x, Mx) \,:\, x \in \mathcal{X}\}$. The function $M \triangleright f$ coincides with $y \mapsto \inf_{x \in \mathcal{X}} F(x, y)$. It is thus convex by Prop. 1.12. □

1.3 Subdifferential

Definition 1.11 (Subdifferential)**.** Let $f : \mathcal{X} \to [-\infty, +\infty]$ and $x \in \operatorname{dom}(f)$. A vector $\phi \in \mathcal{X}$ is called a **subgradient** of f at x if:

$$\forall y \in \mathcal{X}, \quad f(y) \geq f(x) + \langle \phi, y - x \rangle \ .$$

The **subdifferential** of f in x, denoted by $\partial f(x)$, is the set of all the subgradients of f at x. By convention, $\partial f(x) = \emptyset$ if $x \notin \operatorname{dom}(f)$. Formally, $\partial f : \mathcal{X} \to 2^{\mathcal{X}}$ is a set-valued mapping.

Theorem 1.17. *Let $f : \mathcal{X} \to [-\infty, +\infty]$ be a convex function and $x \in \operatorname{ri}(\operatorname{dom} f)$. Then $\partial f(x)$ is non-empty.*

Proof. Let $x_0 \in \operatorname{ri}(\operatorname{dom} f)$. We assume that $f(x_0) > -\infty$ (otherwise the proof is trivial). We may restrict ourselves to the case $x_0 = 0$ and $f(x_0) = 0$ (up to replacing f by the function $x \mapsto f(x + x_0) - f(x_0)$).
In this case, for all vector $\phi \in \mathcal{X}$,

$$\phi \in \partial f(0) \quad \Leftrightarrow \quad \forall x \in \operatorname{dom} f,\ \langle \phi, x \rangle \leq f(x)\,.$$

Let $\mathcal{A} = \operatorname{aff}(\operatorname{dom} f)$. As $\mathcal{A}$ contains point zero, it is an Euclidean vector space.
Let C be the closure of $\operatorname{epi} f \cap (\mathcal{A} \times \mathbb{R})$. The set C is a convex closed set in $\mathcal{A} \times \mathbb{R}$, which is endowed with the scalar product $\langle (x, u), (x', u') \rangle = \langle x, x' \rangle + uu'$.
The point $(0, 0) = (x_0, f(x_0))$ belongs to the boundary of C. Therefore, Th. 1.5 applies in $\mathcal{A} \times \mathbb{R}$. There is a vector $w \in \mathcal{A} \times \mathbb{R}$, $w \neq 0$, such that

$$\forall z \in C,\ \langle w, z \rangle \leq 0$$

Write $w = (\phi, u) \in \mathcal{A} \times \mathbb{R}$. For $z = (x, t) \in C$, we have

$$\langle \phi, x \rangle + u\, t \leq 0.$$

Let $x \in \operatorname{dom}(f)$. In particular $f(x) < \infty$ and for all $t \geq f(x)$, $(x, t) \in C$. Thus,

$$\forall x \in \operatorname{dom}(f),\ \forall t \geq f(x), \quad \langle \phi, x \rangle + u\, t \leq 0. \tag{1.3.1}$$

Letting t tend to $+\infty$, we obtain $u \leq 0$.
Let us prove by contradiction that $u < 0$. Suppose not (*i.e.* $u = 0$). Then $\langle \phi, x \rangle \leq 0$ for all $x \in \operatorname{dom}(f)$. As $0 \in \operatorname{ri}\operatorname{dom}(f)$, there is a set $\tilde{V}$, open in $\mathcal{A}$, such that $0 \in \tilde{V} \subset \operatorname{dom} f$. Thus for $x \in \mathcal{A}$, there is an $\epsilon > 0$ such that $\epsilon x \in \tilde{V} \subset \operatorname{dom}(f)$. According to (1.3.1), $\langle \phi, \epsilon x \rangle \leq 0$, so

$\langle \phi, x \rangle \leq 0$. Similarly, $\langle \phi, -x \rangle \leq 0$. Therefore, $\langle \phi, x \rangle \equiv 0$ on $\mathcal{A}$. Since $\phi \in \mathcal{A}$, $\phi = 0$ as well. Finally $w = 0$, which is a contradiction.
As a result, $u < 0$. Dividing inequality (1.3.1) by $-u$, and taking $t = f(x)$, we get

$$\forall x \in \mathrm{dom}(f),\ \forall t \geq f(x), \quad \langle \frac{-1}{u} \phi, x \rangle \leq f(x)\,.$$

So $\frac{-1}{u} \phi \in \partial f(0)$. □

Definition 1.12. We say that a function f is a **minorant** of a function g if $f \leq g$.

A function $f : \mathcal{X} \to \mathbb{R}$ is affine if there exists $a \in \mathcal{X}$ and $b \in \mathbb{R}$ such that $f(x) = \langle a, x \rangle + b$ for every $x \in \mathcal{X}$.

Proposition 1.18. *Let $f : \mathcal{X} \to (-\infty, +\infty]$ be a convex function. Then, f admits an affine minorant.*

Proof. The result is trivial if f is identically equal to $+\infty$. In the other case (f is proper), $\mathrm{dom}(f) \neq \emptyset$, it holds that $\mathrm{ri}(\mathrm{dom}\, f) \neq \emptyset$ by Th. 1.6. Consider $x_0 \in \mathrm{ri}(\mathrm{dom}\, f)$. By Th. 1.17, there exists $\varphi \in \mathcal{X}$ such that for every $x \in \mathcal{X}$, $f(x) \geq f(x_0) + \langle \varphi, x - x_0 \rangle$. The mapping $x \mapsto f(x_0) + \langle \varphi, x - x_0 \rangle$ is an affine minorant of f. □

When f is differentiable at $x \in \mathrm{dom}\, f$, we denote by $\nabla f(x)$ its gradient at x. The link between differentiation and subdifferential is given by the following proposition :

Proposition 1.19. *Let $f : \mathcal{X} \to (-\infty, \infty]$ be a convex function, differentiable in x. Then $\partial f(x) = \{\nabla f(x)\}$.*

Proof. If f is differentiable at x, the point x necessarily belongs to $\mathrm{int}\big(\mathrm{dom}(f)\big)$. Let $\phi \in \partial f(x)$ and $t \neq 0$. Then for all $y \in \mathrm{dom}(f)$, $f(y) - f(x) \geq \langle \phi, y - x \rangle$. Applying this inequality to $y = x + t(\phi - \nabla f(x))$ (which belongs to $\mathrm{dom}(f)$ for t small enough) leads to :

$$\frac{f\left(x + t(\phi - \nabla f(x))\right) - f(x)}{t} \geq \langle \phi, \phi - \nabla f(x) \rangle\ .$$

The left term converges to $\langle \nabla f(x), \phi - \nabla f(x) \rangle$. Finally,

$$\langle \nabla f(x) - \phi, \phi - \nabla f(x) \rangle \geq 0,$$

i.e. $\phi = \nabla f(x)$. □

Example 1.1. The absolute-value function $x \mapsto |x|$ defined on $\mathbb{R} \to \mathbb{R}$ admits as a subdifferential the sign application, defined by :

$$\mathrm{sign}(x) = \begin{cases} \{1\} & \text{si } x > 0 \\ [-1, 1] & \text{si } x = 0 \\ \{-1\} & \text{si } x < 0\,. \end{cases}$$

1.4 Lower semi-continuity

Definition 1.13 (Reminder : lim inf : **limit inferior**).
The **limit inferior** of a sequence $(u_n)_{n \in \mathbb{N}}$, where $u_n \in [-\infty, \infty]$, is

$$\liminf(u_n) = \sup_{n \geq 0} \Big(\inf_{k \geq n} u_k \Big).$$

Since the sequence $V_n = \inf_{k \geq n} u_k$ is non decreasing, an equivalent definition is

$$\liminf(u_n) = \lim_{n \to \infty} \Big(\inf_{k \geq n} u_k \Big).$$

Definition 1.14 (Lower semicontinuous function)**.** A function $f : \mathcal{X} \to [-\infty, \infty]$ is called **lower semicontinuous (l.s.c.) at** $x \in \mathcal{X}$ if for all sequence (x_n) which converges to x,

$$\liminf f(x_n) \geq f(x).$$

The function f is said to be **lower semicontinuous**, if it is l.s.c. at x, for all $x \in \mathcal{X}$. The function f is said to be **closed**, if $\operatorname{epi}(f)$ is closed.

Proposition 1.20 (epigraphical characterization)**.** *Let $f : \mathcal{X} \to [-\infty, +\infty]$. Then f is l.s.c. if and only if it is closed.*

Proof. If f is l.s.c., and if $(x_n, t_n) \in \operatorname{epi} f \to (\bar{x}, \bar{t})$, then, $\forall n, t_n \geq f(x_n)$. Consequently,

$$\bar{t} = \liminf t_n \geq \liminf f(x_n) \geq f(\bar{x}).$$

Thus, $(\bar{x}, \bar{t}) \in \operatorname{epi} f$, and $\operatorname{epi} f$ is closed.
Conversely, if f is *not* l.s.c., there exists an $x \in \mathcal{X}$, and a sequence $(x_n) \to x$, such that $f(x) > \liminf f(x_n)$, *i.e.*, there is an $\epsilon > 0$ such that $\forall n \geq 0$, $\inf_{k \geq n} f(x_k) \leq f(x) - \epsilon$. Thus, for all n, $\exists k_n \geq k_{n-1}$, $f(x_{k_n}) \leq f(x) - \epsilon$. We have built a sequence $(w_n) = (x_{k_n}, f(x) - \epsilon)$, each term of which belongs to $\operatorname{epi} f$, and which converges to a limit $\bar{w} = (f(x) - \epsilon)$ which is outside the epigraph. Consequently, $\operatorname{epi} f$ is not closed. □

Proposition 1.21. *The sum of two l.s.c. functions is l.s.c.*

Proof. Let $f, g : \mathcal{X} \to [-\infty, +\infty]$ be two l.s.c. functions, and let (x_k) be a sequence converging to $x \in \mathcal{X}$. Then, $\liminf f(x_k) + g(x_k) \geq \liminf f(x_k) + \liminf g(x_k) \geq f(x) + g(x)$. □

Notation. We denote by $\Gamma_0(\mathcal{X})$ the set of all closed proper convex functions on $\mathcal{X} \to (-\infty, +\infty]$.

1.5 Minimizers, coercivity, strict and strong convexity

A point x is called a **minimizer** of f if $f(x) \leq f(y)$ for all $y \in \mathcal{X}$. The set of minimizers of f is denoted $\arg\min(f)$.

Proposition 1.22 (Fermat's rule)**.** $x \in \arg\min f \;\Leftrightarrow\; 0 \in \partial f(x)$.

Proof. This follows from the definition: $0 \in \partial f(x) \;\Leftrightarrow\; \forall y, f(y) \geq f(x) + \langle 0, y - x\rangle$. This means that $x \in \arg\min f$. □

Definition 1.15. A function $f : \mathcal{X} \to (-\infty, +\infty]$ is said coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$.

Otherwise stated, f is coercive if, for every sequence (x_n) such that $\|x_n\| \to +\infty$, it holds that $f(x_n) \to +\infty$.

Proposition 1.23. *Let $f : \mathcal{X} \to (-\infty, +\infty]$ be coercive and l.s.c. Then, f admits a minimizer.*

Proof. We consider the case where f is not identically equal to $+\infty$ (otherwise the proof is trivial). Consider a sequence (x_n) such that $f(x_n) \to \inf f(\mathcal{X})$. Such a sequence exists by definition of the infimum. The sequence is bounded. Indeed, if it were unbounded, one would be able to extract a subsequence (x_{φ_n}) converging to $+\infty$ in norm. The coercivity assumption would imply that $f(x_{\varphi_n}) \to +\infty$ which would contradict the fact that $f(x_n) \to \inf f(\mathcal{X})$. As (x_n) is bounded, one can extract a converging subsequence, say $x_{\psi_n} \to x^*$ for some $x^* \in \mathcal{X}$. As f is l.s.c., $\inf f(\mathcal{X}) = \lim f(x_{\psi_n}) = \liminf f(x_{\psi_n}) \geq f(x^*)$. Thus, $f(x^*) = \inf f(\mathcal{X})$ and x^* is a minimizer. □

Definition 1.16. A function $f : \mathcal{X} \to (-\infty, +\infty]$ is said **strictly convex** if for every $x, y \in \operatorname{dom} f$ s.t. $x \neq y$, and every $t \in (0,1)$, $f(tx + (1-t)y) < tf(x) + (1-t)f(y)$.

Proposition 1.24. *A stricly convex function $f : \mathcal{X} \to (-\infty, +\infty]$ with non empty domain admits at most one minimizer.*

Proof. By contradiction, consider two distinct minimizers x, y. Then, $f((x+y)/2) < (f(x) + f(y))/2 = \inf f(\mathcal{X})$, which is impossible. □

Example 1.2. The squared Euclidean norm $x \mapsto \|x\|^2$ is strictly convex.

Definition 1.17. Let $\mu > 0$. A function $f : \mathcal{X} \to (-\infty, +\infty]$ is said **μ-strongly convex** if $f - \frac{\mu}{2}\|\,.\,\|^2$ is convex. It is said strongly convex if it is μ-strongly convex for some $\mu > 0$.

Proposition 1.25. *Any strongly convex function $f : \mathcal{X} \to (-\infty, +\infty]$ is strictly convex and coercive. As a consequence, a strongly convex function in $\Gamma_0(\mathcal{X})$ admits a unique minimizer.*

Proof. We only consider the case where $f \not\equiv +\infty$, otherwise the first result is trivial. Set $g = f - \frac{\mu}{2}\|\,.\,\|^2$. By Prop. 1.18, the convex function g admits an affine minorant, say h. Thus, $f \geq h + \frac{\mu}{2}\|\,.\,\|^2$. As h is affine, it holds that $h(x) + \frac{\mu}{2}\|x\|^2$ tends to $+\infty$ as $\|x\| \to +\infty$. Hence, f is coercive. Consider $x \neq y$ in $\operatorname{dom} f$, and $t \in (0,1)$. By convexity of g, $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$. Therefore,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \frac{\mu}{1}\left(\|tx + (1-t)y\|^2 - t\|x\|^2 - (1-t)\|y\|^2\right) .$$

The term enclosed in the parenthesis is striclty negative by the strict convexity of the squared norm (see Example 1.2). Thus, f is strictly convex. The second result follows from Propositions 1.23 and 1.24. □

1.6 Exercises

Exercise 1.1. Let C be a subset of $\mathbb{R}^d$ with a non-empty interior. Prove that $\operatorname{ri}(C) = \operatorname{int}(C)$.

Exercise 1.2. Let $f : \mathcal{X} \to [-\infty, \infty]$, and assume that $\partial f(x)$ and $\partial f(y)$ are non empty for some $x, y \in \mathcal{X}$. Show that

$$\forall u \in \partial f(x),\ \forall v \in \partial f(y),\ \langle x - y, u - v\rangle \geq 0\,.$$

Exercise 1.3. Consider m Euclidean spaces $\mathcal{X}_1, \cdots, \mathcal{X}_m$, where $m \geq 1$ is an integer. For every $i \in \{1, \cdots, m\}$, let $f_i : \mathcal{X}_i \to (-\infty, +\infty]$ be a function and consider the function $f : \prod_{i=1}^m \mathcal{X}_i \to (-\infty, +\infty]$ such that

$$f : (x_1, \cdots, x_m) \mapsto \sum_{i=1}^m f_i(x_i)\,.$$

1. Prove that

$$\inf f = \sum_{i=1}^m \inf f_i\,.$$

2. Prove that

$$\arg\min f = \prod_{i=1}^m \arg\min f_i\,.$$

3. Prove that
$$\partial f = \prod_{i=1}^{m} \partial f_i \,.$$

Exercise 1.4. Prove that the set $\{x \in \mathcal{X} : Ax = b\}$ is affine, where A is a linear operator on $\mathcal{X} \to \mathcal{Y}$ and $b \in \mathcal{Y}$. Conversely, prove that any affine space can be written under this form.

Exercise 1.5. Consider a function $f : \mathcal{X} \to (-\infty, +\infty]$ and let $\phi \in \mathcal{X}$. Prove that the subdifferential of the function $x \mapsto f(x) + \langle \phi, x\rangle$ coincides, at a given point $x \in \mathcal{X}$, with $\partial f(x) + \phi$.

Exercise 1.6 (normal cone)**.** For every set $C \subset \mathcal{X}$ and every $x \in C$, we define $N_C(x) = \{\varphi \in \mathcal{X} : \forall y \in C,\ \langle \varphi, y - x\rangle \leq 0\}$. We set $N_C(x) = \emptyset$ for every $x \notin C$.

1. Prove the identity $N_C = \partial \iota_C$.
2. Show that for all $x \in C$, $N_C(x)$ is a cone, *i.e.* $\forall \varphi \in N_C(x)$, $\forall \lambda \geq 0$, $\lambda\varphi \in N_C(x)$. It is called the **normal cone** of C at x.
3. Prove that $N_C(x) = \{0\}$ whenever $x \in \mathrm{int}(C)$.
4. Derive $N_C(x)$ in the following cases: $C = (-\infty, 0]$, $C = (-\infty, 1]$, $C = \mathbb{R}_- \times \mathbb{R}_-$, $C = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\}$.

Exercise 1.7. Let C and D be two convex subsets of $\mathcal{X}$. Prove that $\mathrm{ri}(C - D) = \mathrm{ri}(C) - \mathrm{ri}(D)$. *Hint: apply Th. 1.7 to the convex set $C \times D$.*

Exercise 1.8. Show that if f is convex, proper, with $\mathrm{dom}\, f = \mathcal{X}$, and if f is bounded, then f is constant.

Exercise 1.9 (the question of $-\infty$ values)**.**
Let $f : \mathcal{X} \to [-\infty, +\infty]$ be convex and assume that $\mathrm{ri}\,\mathrm{dom}\, f$ contains a point x such that $f(x) > -\infty$. Prove that f *never* takes the value $-\infty$. Thus, f is proper.
(Hint: use Prop. 1.17).

Exercise 1.10. Determine the subdifferentials of the following functions, at the considered points:

1. In $\mathcal{X} = \mathbb{R}$, $f(x) = \iota_{[0,1]}$, at $x = 0, x = 1$ and $0 < x < 1$.
2. In $\mathcal{X} = \mathbb{R}^2$, $f(x_1, x_2) = \iota_{x_1 < 0}$, at x such that $x_1 = 0$, $x_1 < 0$.
3. $\mathcal{X} = \mathbb{R}$,
$$f(x) = \begin{cases} +\infty & \text{si } x < 0 \\ -\sqrt{x} & \text{si } x \geq \end{cases}$$
at $x = 0$, and $x > 0$.
4. $\mathcal{X} = \mathbb{R}^n$, $f(x) = \|x\|$, determine $\partial f(x)$, for any $x \in \mathbb{R}^n$.
5. $\mathcal{X} = \mathbb{R}$, $f(x) = x^3$. Show that $\partial f(x) = \emptyset$, $\forall x \in \mathbb{R}$. Explain this result.
6. $\mathcal{X} = \mathbb{R}^n$, $C = \{y : \|y\| \leq 1\}$, $f(x) = \iota_C(x)$. Give the subdifferential of f at x such that $\|x\| < 1$ and at x such that $\|x\| = 1$.

   Hint: For $\|x\| = 1$:

   - Show that $\partial f(x) = \{\phi :\ \forall y \in C, \langle \phi, y - x\rangle \leq 0.$

- Show that $x \in \partial f(x)$ using Cauchy-Schwarz inequality. Deduce that the cone $\mathbb{R}^+ x = \{t\,x \;:\; t \geq 0\} \subset \partial f(x)$.
- To show the converse inclusion : Fix $\phi \in \partial f$ and pick $u \in \{x\}_\perp$ (*i.e.*, u s.t. $\langle u, x\rangle = 0$). Consider the sequence $y_n = \|x + t_n u\|^{-1}(x + t_n u)$, for some sequence $(t_n)_n, t_n > 0, t_n \to 0$. What is the limit of y_n ?

  Consider now $u_n = t_n^{-1}(y_n - x)$. What is the limit of u_n ? Conclude about the sign of $\langle \phi, u\rangle$.

  Do the same with $-u$, conclude about $\langle \phi, u\rangle$. Conclude.

7. Let $f : \mathbb{R} \to \mathbb{R}$, differentiable. Show that: f is convex, if and only if

$$\forall (x, y) \in \mathbb{R}^2, \langle \nabla f(y) - \nabla f(x), y - x\rangle \geq 0.$$

Exercise 1.11. Show that a function f is l.s.c. if and only if its level sets :

$$L_{\leq \alpha} = \{x \in \mathcal{X} : f(x) \leq \alpha\}$$

are closed.
(see, *e.g.*, Rockafellar and Wets (1998), theorem 1.6.)

Exercise 1.12. Let $f : \mathcal{X} \to [-\infty, +\infty]$ be l.s.c. and convex. Assume that $-\infty \in f(\mathcal{X})$. Then $f(\mathcal{X}) \subset \{-\infty, +\infty\}$.

Chapter 2

Fenchel-Legendre transform

2.1 Definitions and properties

Definition 2.1. Let $f : \mathcal{X} \to [-\infty, +\infty]$. The **Fenchel-Legendre transform**, or Fenchel-Legendre conjugate, of f is the function $f^* : \mathcal{X} \to [-\infty, \infty]$, defined by

$$f^*(\phi) = \sup_{x \in \mathcal{X}} \langle \phi, x \rangle - f(x) \,, \qquad \phi \in \mathcal{X}.$$

The following lemma is straightforward.

Lemma 2.1. *Let $f, g : \mathcal{X} \to [-\infty, +\infty]$. If $f \leq g$ then $f^* \geq g^*$.*

Proposition 2.2. *Let $f : \mathcal{X} \to [-\infty, +\infty]$. Then f^* is convex and l.s.c.*

Proof. Remark that f^* is the upper hull of the collection of functions $(\phi \mapsto \langle \phi, x \rangle - f(x) : x \in \mathcal{X})$. Each of these functions is affine, it is thus l.s.c. and convex. The result follows from Prop.1.11. □

Remark 2.1. By definition of f^*, it is clear that $-\infty \in f^*(\phi)$ iff $f \equiv +\infty$. In other words, f^* never takes the value $-\infty$ unless f is identically $+\infty$.

For every $\phi \in \mathcal{X}$, we denote by $\mathcal{A}_\phi(f)$ the set of affine minorants of gradient ϕ, that is:

$$\mathcal{A}_\phi(f) := \{\alpha \,:\, \alpha : \mathcal{X} \to \mathbb{R} \text{ is an affine minorant of } f \text{ and } \nabla\alpha \equiv \phi\} \,.$$

Note that every $\alpha \in \mathcal{A}_\phi(f)$ has the form $\alpha(x) = \langle \phi, x \rangle + \alpha(0)$ for every $x \in \mathcal{X}$. If there exists $\alpha \in \mathcal{A}_\phi(f)$ such that for every $\beta \in \mathcal{A}_\phi(f)$, $\alpha \geq \beta$, we say that α is the **maximal element** of $\mathcal{A}_\phi(f)$. If such a maximal element exists, it is unique.

Proposition 2.3. *Let $f : \mathcal{X} \to (-\infty, +\infty]$ be proper and let $\phi \in \mathcal{X}$. Then,*

1. *If $f^*(\phi) = +\infty$, then f admits no affine minorant of gradient ϕ (*i.e.*, $\mathcal{A}_\phi(f) = \emptyset$).*
2. *If $f^*(\phi) \in \mathbb{R}$, the function $x \mapsto \langle \phi, x \rangle - f^*(\phi)$ is the maximal element in $\mathcal{A}_\phi(f)$.*

Proof. For every $\alpha \in \mathcal{A}_\phi(f)$, $\alpha(x) = \langle \phi, x \rangle + \alpha(0)$ for every $x \in \mathcal{X}$. As $\alpha \leq f$, it holds that $-\alpha(0) \geq \langle \phi, x \rangle - f(x)$ for every $x \in \mathcal{X}$. Hence, $-\alpha(0) \geq f^*(\phi)$. This cannot happen if $f^*(\phi) = +\infty$, and the first point is proved. For every $x \in \mathcal{X}$, the former inequality implies that $\langle \phi, x \rangle - f^*(\phi) \geq \langle \phi, x \rangle + \alpha(0)$. This proves that $\langle \phi, . \rangle - f^*(\phi) \geq \alpha$ for every $\alpha \in \mathcal{A}_\phi(f)$, and the second point is proved. □

The set

$$\mathcal{A}(f) = \bigcup_{\phi \in \mathcal{X}} \mathcal{A}_\phi(f)$$

is the set of affine minorants of f. Recall that this set is empty for every function $f : \mathcal{X} \to (-\infty, +\infty]$ that is convex. The upper hull of $\mathcal{A}(f)$ is called the **affine enveloppe** of f *i.e.*, it is the function

$$x \mapsto \sup\{\alpha(x) \,:\, \alpha \in \mathcal{A}\}\,,$$

where we recall the convention $\sup \emptyset = -\infty$. We use the notation f^{**} to designate $(f^*)^*$. The function f^{**} is called the **biconjugate**, and satisfies for every $x \in \mathcal{X}$,

$$f^{**}(x) = \sup_{\phi \in \mathcal{X}} \langle \phi, x\rangle - f^*(\phi)\,.$$

Proposition 2.4. *For every $f : \mathcal{X} \to [-\infty, +\infty]$, f^{**} is the affine enveloppe of f. In particular, $f \geq f^{**}$.*

Proof. Denote by h the affine enveloppe of f. Assume that f is proper. Consider $\alpha \in \mathcal{A}(f)$. There exists ϕ such that $\alpha \in \mathcal{A}_\phi(f)$. By Prop. 2.3, $\alpha(x) \leq \langle \phi, x\rangle - f^*(\phi)$ for all $x \in \mathcal{X}$. Hence, $\alpha(x) \leq f^{**}(x)$. Taking the supremum w.r.t. $\alpha \in \mathcal{A}$, $h(x) \leq f^{**}(x)$. To show the other inequality, note that for every $\phi \in \mathrm{dom}(f^*)$, $\langle \phi, \,.\,\rangle - f^*(\phi) \in \mathcal{A}(f)$ by Prop. 2.3. Thus, for every $x \in \mathcal{X}$,

$$f^{**}(x) = \sup\{\langle \phi, x\rangle - f^*(\phi) \,:\, \phi \in \mathrm{dom}(f^*)\} \leq h(x)\,.$$

This shows that $f^{**} = h$.
When $f \equiv +\infty$, it is clear that $h \equiv +\infty$. We have $f^* \equiv -\infty$ which in turn yields $f^{**} \equiv +\infty$. Thus, $f^{**} = h$.
When $-\infty \in f(\mathcal{X})$, f has no affine minorant and $h \equiv -\infty$. We have $f^* \equiv +\infty$ which implies $f^{**} \equiv -\infty$. Thus, $f^{**} = h$ as well.
The fact that $f \geq f^{**}$ is straightforward from the definition of the affine enveloppe. □

2.2 The Fenchel-Moreau theorem

Theorem 2.5 (Fenchel-Moreau)**.** *Consider $f : \mathcal{X} \to (-\infty, +\infty]$. Then $f = f^{**}$ if and only if f is convex and l.s.c.*

Proof. By Prop. 2.2, f^{**} is always convex and l.s.c. Therefore, $f = f^{**}$ implies that f is convex and l.s.c. We prove the converse. By definition, the affine enveloppe minorates f. Thus, by Prop. 2.3, $f^{**} \leq f$. We now show that $f^{**} \geq f$. The result is trivial when $f \equiv +\infty$, we thus consider that f is proper. By contradiction, assume that there exists x_0 such that $f^{**}(x_0) < f(x_0)$. Clearly, the point $z_0 := (x_0, f^{**}(x_0))$ is such that $z_0 \notin \mathrm{epi}(f)$. As f is proper, l.s.c. and convex, $\mathrm{epi}(f)$ is a non-empty, closed and convex set. By Prop. 1.4, there exists a non-zero vector $(\phi, u) \in \mathcal{X} \times \mathbb{R}$ and $a \in \mathbb{R}$ such that $\langle w, z\rangle + a \leq 0$ for every $z \in \mathrm{epi}(f)$ and $\langle w, z_0\rangle + a > 0$. This reads:

$$\forall x \in \mathrm{dom}(f), \forall t \geq f(x),\; \langle \phi, x\rangle + ut + a \leq 0\,, \tag{2.2.1}$$

$$\text{and}\;\; \langle \phi, x_0\rangle + uf^{**}(x_0) + a > 0\,, \tag{2.2.2}$$

Letting $t \to +\infty$ in (2.2.1), it follows that $u \leq 0$.
First consider the case where $u < 0$. Putting $t = f(x)$ in (2.2.1) and dividing both sides of the inequality by $-u$, it follows that for every $x \in \mathrm{dom}(f)$,

$$\left\langle -\tfrac{\phi}{u}, x\right\rangle + \tfrac{a}{-u} \leq f(x)\,.$$

Therefore, the affine map $\langle -\phi/u, x\rangle - a/u$ lies in $\mathcal{A}(f)$. By Prop. 2.4, $f^{**}(x_0) \geq \langle -\phi/u, x_0\rangle - a/u$ for every x. This contradicts (2.2.2).
Now consider the case where $u = 0$. Assume that $f \geq 0$. Eq. (2.2.1) reads $\langle \phi, x\rangle + a \leq 0$ for every $x \in \mathrm{dom}(f)$. Choose $\varepsilon > 0$. In particular, $\langle \phi, x\rangle + a \leq \varepsilon f(x)$, because of the assumption that $f \geq 0$. Therefore, $\langle \phi/\varepsilon, \,.\,\rangle + a/\varepsilon \in \mathcal{A}(f)$. By Prop. 2.4, $f^{**}(x_0) \geq \langle \phi/\varepsilon, x\rangle + \frac{a}{\varepsilon}$. As a consequence, $\varepsilon f^{**}(x_0) \geq \langle \phi, x_0\rangle + a$. Letting $\varepsilon \to 0$, we obtain that $0 \geq \langle \phi, x_0\rangle + a$, which contradicts (2.2.2).
We have thus shown that if f is convex, l.s.c. and non negative, $f = f^{**}$. Now consider the case where f is convex, l.s.c., but may take negative values. By Prop. 1.18, f admits an affine minorant, say α. The function $f - \alpha$ is convex, l.s.c. and non negative, and thus satisfies $f - \alpha = (f - \alpha)^{**}$. The proof is concluded upon noting that $(f - \alpha)^{**} = f^{**} - \alpha$ which follows from straightforward algebra. □

Definition 2.2. Let $f : \mathcal{X} \to [-\infty, +\infty]$. The function $\breve{f} : \mathcal{X} \to [-\infty, +\infty]$ defined for every $x \in \mathcal{X}$ by

$$\breve{f}(x) = \sup\{g(x) \,:\, g : \mathcal{X} \to [-\infty, +\infty] \text{ is a convex and l.s.c. minorant of } f\},$$

is called the l.s.c. convex envelope of f.

Lemma 2.6. *For every $f : \mathcal{X} \to [-\infty, +\infty]$, $\breve{f}$ is l.s.c. and convex.*

Proof. As an exercise. □

Lemma 2.7. *Consider a convex function $f : \mathcal{X} \to [-\infty, +\infty]$ and a point $x \in \mathcal{X}$. Then, f is l.s.c. at x if and only if $f(x) = \breve{f}(x)$.*

Proof. i) Assume that $f(x) = \breve{f}(x)$. Choose a sequence $x_n \to x$. Set $\varepsilon > 0$. By definition of the supremum, there exists a convex l.s.c. function g such that $g(x) \geq f(x) - \varepsilon$. As g is l.s.c., $\liminf_n g(x_n) \geq g(x)$. As g is a minorant of f, it holds moreover that $g(x_n) \leq f(x_n)$. Putting all pieces together, $\liminf_n f(x_n) \geq f(x) - \varepsilon$. As ε is arbitrary, this implies that $\liminf_n f(x_n) \geq f(x)$, hence f is l.s.c. at x.
ii) Assume that f is l.s.c. at x. The inequality $\breve{f}(x) \leq f(x)$ is immediate from the definition of $\breve{f}$. Thus, we only need to prove that $\breve{f}(x) \geq f(x)$. Introduce the function g defined for every $x' \in \mathcal{X}$ by

$$g(x') = \lim_{n\to\infty} \inf_{y:\|y-x'\|\leq\frac{1}{n}} f(y)\,. \tag{2.2.3}$$

It is clear that $g \leq f$. Moreover, we shall establish below that g is convex and l.s.c. Therefore, by the definition of $\breve{f}$, it holds that $\breve{f}(x) \geq g(x)$. Using the fact that f is l.s.c. at x, we have that $f(x) \leq g(x)$. Indeed, for every $\varepsilon > 0$, for every n, there exists y_n such that $\|y_n - x\| \leq \frac{1}{n}$ and $f(y_n) \leq \inf_{y:\|y-x'\|\leq\frac{1}{n}} f(y) + \varepsilon$. Thus, $\liminf f(y_n) \leq g(x) + \varepsilon$. As f is l.s.c. at x, this shows that $f(x) \leq g(x) + \varepsilon$. As ε is arbitrary, this implies that $f(x) \leq g(x)$. Recalling that $\breve{f}(x) \geq g(x)$, we conclude that $\breve{f}(x) \geq f(x)$, which is the desired conclusion.
We now establish that g is convex. Remark that the limit in Eq.(2.2.3) can be replaced by a supremum w.r.t. n, because the sequence of infimum is increasing. Therefore $g(x') = \sup_n F_n(x')$ where we define $F_n(x') = \inf_y(f(y) + \iota_{\Delta_n}(x', y))$, where $\Delta_n = \{(x', y) : \|x' - y\| \leq \frac{1}{n}\}$ is a convex set. For every n, the function F_n is convex by Prop. 1.12. Thus, g is convex as an upper hull of convex functions.
We finally establish that g is l.s.c. Consider a sequence $(x_k, t_k) \in \mathrm{epi}(g)$ such that (x_k, t_k) converges to some point (x', t) as $k \to \infty$. Set $\varepsilon > 0$. For every k, there exists y_k such that $\|y_k - x_k\| \leq \frac{1}{k}$ and such that $f(y_k) \leq g(x_k) + \varepsilon$. Therefore, $t_k \geq f(y_k) - \varepsilon$. Thus, $t \geq \liminf_k f(y_k) - \varepsilon$. As $y_k \to x'$, $\liminf_k f(y_k) \geq g(x')$ As ε is arbitrary, it follows that $t \geq g(x')$. Hence, $(x', t)\,\mathrm{epi}(g)$. This means that $\mathrm{epi}(g)$ is closed. □

Theorem 2.8. *Consider a convex function $f : \mathcal{X} \to (-\infty, +\infty]$ and a point $x \in \mathcal{X}$. Then, $f(x) = f^{**}(x)$ if and only if f is l.s.c. at x.*

Proof. We prove that $\breve{f} = f^{**}$ and the result follows from lem. 2.7.
Clearly the set of convex and l.s.c. minorants of f contains $\mathcal{A}(f)$. This implies that $\breve{f} \geq f^{**}$ *i.e.*, the l.s.c. convex envelope majorizes the affine envelope. We prove the other inequality. By definition of the l.s.c. convex envelope, $\breve{f} \leq f$. By Lemma 2.1, $\breve{f}^* \geq f^*$ and thus $\breve{f}^{**} \leq f^{**}$. By Prop. 1.18, f admits an affine minorant, and thus admits a real valued l.s.c. and convex minorant. Therefore, $-\infty \notin \breve{f}(\mathcal{X})$. Lem. 2.6 and Th. 2.5 together imply that $\breve{f} = \breve{f}^{**}$. Thus, $\breve{f} \leq f^{**}$. Hence, $\breve{f} = f^{**}$. □

2.3 The Fenchel-Young inequality and its consequences

Proposition 2.9 (Fenchel - Young)**.** *Let $f : \mathcal{X} \to [-\infty, \infty]$. For all $(x, \phi) \in \mathcal{X}^2$, the following inequality holds:*

$$f(x) + f^*(\phi) \geq \langle \phi, x \rangle ,$$

with equality if and only if $\phi \in \partial f(x)$.

Proof. The inequality is an immediate consequence of the definition of f^*. The condition for equality to hold (*i.e.*, for the converse inequality to be valid), is obtained with the equivalence

$$f(x) + f^*(\phi) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \forall y,\; f(x) + \langle \phi, y \rangle - f(y) \leq \langle \phi, x \rangle \;\Leftrightarrow\; \phi \in \partial f(x) .$$

□

Proposition 2.10. *Let $f : \mathcal{X} \to (-\infty, \infty]$ be proper, convex, and l.s.c. at some point $x \in \mathcal{X}$. Then,*

$$\phi \in \partial f(x) \;\Leftrightarrow\; x \in \partial f^*(\phi) .$$

Proof. By the equality case in Prop. 2.9, $\phi \in \partial f(x)$ iff $f(x) + f^*(\phi) = \langle \phi, x \rangle$. Since f is l.s.c. at $x \in \mathcal{X}$, Th. 2.8 implies that $f(x) = f^{**}(x)$. Thus, $\phi \in \partial f(x)$ iff $f^{**}(x) + f^*(\phi) = \langle \phi, x \rangle$. By Prop. 2.9 again, this is equivalent to $x \in \partial f^*(\phi)$. □

If $A : \mathcal{X} \to 2^{\mathcal{X}}$ is a set-valued mapping, we define $A^{-1} : \mathcal{X} \to 2^{\mathcal{X}}$ the **inverse** of A as

$$A^{-1}(x) = \{y \,:\, x \in A(x)\} ,$$

namely, $y \in A^{-1}(x) \Leftrightarrow x \in A(y)$.

Corollary 2.11. *Let $f \in \Gamma_0(\mathcal{X})$. Then, $\partial f^{-1} = \partial f^*$.*

Proof. The result follows from Prop. 2.10 and the Fenchel-Moreau theorem Th. 2.5. □

Theorem 2.12. *Consider $f : \mathcal{X} \to (-\infty, +\infty]$ a proper and convex function. Let $x \in \mathcal{X}$. Consider the following assertions:*

i) $x \in \mathrm{ri}(\mathrm{dom}\, f)$,

ii) $\partial f(x) \neq \emptyset$,

iii) *f is l.s.c. at x* ,

iv) $f(x) = f^{**}(x)$.

v) $\partial f(x) = \partial f^{**}(x)$.

Then, i) $\Rightarrow$ *ii)* $\Rightarrow$ *iii)* $\Leftrightarrow$ *iv)* $\Rightarrow$ *v).*

Proof. *i)* $\Rightarrow$ *ii)* is given by Th. 1.17. *iii)* $\Leftrightarrow$ *iv)* is given by Th. 2.8.
We prove that *ii)* $\Rightarrow$ *iv)*. Choose $\phi \in \partial f(x)$. By the equality case in the Fenchel-Young inequality (Prop. 2.9), $f(x) = \langle \phi, x\rangle - f^*(\phi)$. Thus, $f(x) \leq \sup_{\psi \in \mathcal{X}} \langle \psi, x\rangle - f^*(\psi)$ and the righthand side coincides with $f^{**}(x)$. Thus, $f(x) \leq f^{**}(x)$. On the otherhand $f^{**} \leq f$ since f^{**} is the affine envelope of f (Prop. 2.4). Thus $f(x) = f^{**}(x)$.
Finally, we prove that *iv)* $\Rightarrow$ *v)*. Consider $\phi \in \mathcal{X}$. By Prop. 2.9, $\phi \in \partial f(x) \Leftrightarrow f(x) + f^*(\phi) = \langle \phi, x\rangle$. By the standing hypothesis, this rewrites as $f^{**}(x) + f^*(\phi) = \langle \phi, x\rangle$. Since f^* is convex, l.s.c., and does not take the value $-\infty$ (as f is proper), Th. 2.5 implies that $f^* = f^{***}$. Therefore, the following equivalence holds: $\phi \in \partial f(x) \Leftrightarrow f^{**}(x) + f^{***}(\phi) = \langle \phi, x\rangle$. By Prop. 2.9, this is equivalent to $\phi \in \partial f^{**}(\phi)$. □

Proposition 2.13. *If f is μ-strongly convex, then f^* is differentiable and ∇f^* is μ^{-1}-Lipschitz continuous.*

Proof. Let $\lambda \in \mathcal{Y}$. By the Fenchel-Young property 2.9, $x \in \partial f^*(\lambda)$ if and only if $\lambda \in \partial f(x)$. By Fermat's rule, this is again equivalent to $x \in \arg\min f - \langle \lambda, .\rangle$. As f is strongly convex, the argument of the minimum exists and is unique. As such, x is well and uniquely defined. Thus, $\partial f^*(\lambda)$ is a singleton. Let us denote $\partial f^*(\lambda) = \{\nabla f^*(\lambda)\}$. We shall show later that f^* is indeed differentiable.
We first verify the Lipschitz continuity of ∇f^*. Let us fix λ and λ'. Set $x = \nabla f^*(\lambda)$ and $y = \nabla f^*(\lambda')$. Since $\lambda \in \partial f(x)$, the strong convexity of f implies

$$f(y) \geq f(x) + \langle \lambda, y - x\rangle + \frac{\mu}{2}\|y - x\|^2$$

and a similar inequality hold by symetry:

$$f(x) \geq f(y) + \left\langle \lambda', x - y\right\rangle + \frac{\mu}{2}\|y - x\|^2$$

Summing these inequalities leads to $0 \geq \langle \lambda - \lambda', y - x\rangle + \mu\|x - y\|^2$. Hence, $\|y - x\|^2 \leq \frac{1}{\mu}\|\lambda - \lambda'\|\,\|x - y\|$ by the Cauchy-Schwarz inequality. Thus $\|y - x\| \leq \frac{1}{\mu}\|\lambda - \lambda'\|$ which means that $\|\nabla f^*(\lambda') - \nabla f^*(\lambda)\| \leq \frac{1}{\mu}\|\lambda - \lambda'\|$.
We finish with the differentiability of f^*. Let $x, h \in \mathcal{X}$. Since $\nabla f^*(x) \in \partial f^*(x)$ and $\nabla f^*(x+h) \in \partial f^*(x + h)$,

$$f^*(x + h) \geq f^*(x) + \langle \nabla f^*(x), h\rangle \quad \text{and} \quad f^*(x + h) \leq f^*(x) + \langle \nabla f^*(x + h), h\rangle$$

We use the Lipschitz continuity of ∇f^* to get $\nabla f^*(x + h) = \nabla f^*(x) + \epsilon$ where $\|\epsilon\| \leq \frac{1}{\mu}\|h\|$. This leads to $f^*(x+h) \leq f^*(x) + \langle \nabla f^*(x), h\rangle + \frac{1}{\mu}\|h\|^2$ and so, with the other inequality, we get $f^*(x + h) = f^*(x) + \langle \nabla f^*(x), h\rangle + o(h)$. □

Remark 2.2. Lemma 2.13 has a converse. We refer to (Hiriart-Urruty and Lemaréchal, 2012, Theorem 4.2.2).

2.4 Exercises

Exercise 2.1 (invariance by Fenchel-Legendre conjugation)**.** Let $f : \mathcal{X} \to [-\infty, +\infty]$. Show that $f = f^* \Leftrightarrow f(x) = \frac{1}{2}\|x\|^2$.

Exercise 2.2. Set $\mathcal{X} := \mathbb{R}^m$ where $m \geq 1$ is an integer. Prove that the Fenchel conjugate of $\iota_{(-\infty,0]^m}$ is equal to $\iota_{[0,+\infty)^m}$.

Exercise 2.3. Let E be a linear subspace of $\mathcal{X}$ and denote by $E^\perp$ its supplementary set. Prove that the Fenchel conjugate of ι_E is equal to $\iota_{E^\perp}$.

Exercise 2.4.

1. On $\mathcal{X} := \mathbb{R}$, define

$$f(x) = \begin{cases} 1/x & \text{if } x > 0; \\ +\infty & \text{otherwise .} \end{cases}$$

Prove that

$$f^*(\phi) = \begin{cases} -2\sqrt{-\phi} & \text{if } \phi \leq 0; \\ +\infty & \text{otherwise .} \end{cases}$$

2. On $\mathcal{X} := \mathbb{R}$, define $f(x) = \exp(x)$. Prove that

$$f^*(\phi) = \begin{cases} \phi \ln(\phi) - \phi & \text{if } \phi > 0; \\ 0 & \text{if } \phi = 0; \\ +\infty & \text{if } \phi < 0. \end{cases}$$

Exercise 2.5 (Fenchel-Legendre transform of an infimal post composition)**.** Consider $f : \mathcal{X} \to (-\infty, +\infty]$ and let $M : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Prove that for every $\phi \in \mathcal{Y}$, $(M \triangleright f)^*(\phi) = f^*(M^*\phi)$.

Exercise 2.6 (Fenchel-Legendre transform of an infimal convolution)**.** Let $f, g : \mathcal{X} \to (-\infty, +\infty]$ be proper convex functions.

1. Assume that f and g both admit an affine minorant with gradient ϕ. Show that $f \,\square\, g$ has an affine minorant of gradient ϕ.

2. Prove that $(f \,\square\, g)^* = f^* + g^*$.

Chapter 3

Duality

3.1 Parametric Duality

In this paragraph, we consider a function $F : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]$ where $\mathcal{X}$ and $\mathcal{Y}$ are two Euclidean spaces. Without risk of ambiguity, we can denote with the same symbol $\langle\, .\,,\, .\,\rangle$ the scalar products in each of these spaces. We equipp the product space $\mathcal{X} \times \mathcal{Y}$ with the scalar product $\langle(\nu, \phi), (x, y)\rangle := \langle \nu, x\rangle + \langle \phi, y\rangle$.

3.1.1 Primal and dual problems

The **primal function** and the **dual function** are the mappings $\mathcal{P} : \mathcal{X} \to [-\infty, +\infty]$ and $\mathcal{D} : \mathcal{Y} \to [-\infty, +\infty]$ respectively defined by

$$\begin{aligned} \mathcal{P}(x) &= F(x, 0) \quad (\forall x \in \mathcal{X}) \\ \mathcal{D}(\phi) &= -F^*(0, -\phi) \quad (\forall \phi \in \mathcal{Y})\,. \end{aligned}$$

We respectively refer to $p := \inf \mathcal{P}$ and $d := \sup \mathcal{D}$ as the **primal (resp. dual) value** and to $\arg\min \mathcal{P}$ and $\arg\max \mathcal{D}$ as the set of **primal (resp. dual) solutions**. Finally, the **value function** is the mapping $\vartheta : \mathcal{Y} \to [-\infty, +\infty]$ defined for all $y \in \mathcal{Y}$ by

$$\vartheta(y) := \inf F(\mathcal{X}, y)\,.$$

Lemma 3.1. *The following holds:*

1. $p = \vartheta(0)$ *and* $d = \vartheta^{**}(0)$.
2. *For every* $\phi \in \mathcal{Y}$, $\mathcal{D}(\phi) = -\vartheta^*(-\phi)$.
3. *If* ϑ^* *is proper,* $\arg\max \mathcal{D} = -\partial\vartheta^{**}(0)$. *Otherwise,* $\arg\max \mathcal{D} = \mathcal{Y}$ *and* $d \in \{-\infty, +\infty\}$.

Proof. The equality $p = \vartheta(0)$ is obvious from the definition. For every $\phi \in \mathcal{Y}$,

$$\vartheta^*(\phi) = \sup_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \langle \phi, y\rangle - F(x, y) = F^*(0, \phi)\,,$$

and the second point is proved. The fact that $d = \vartheta^{**}(0)$, follows from the definition of $\vartheta^{**}(0) = \sup_{\phi\in\mathcal{Y}} -\vartheta^*(\phi)$. The first point is proved. This also shows that the dual solutions are the opposite of the minimizers of ϑ^*: $\arg\max \mathcal{D} = -\arg\min \vartheta^*$. If ϑ is proper, then

$$\begin{aligned}\phi \in \arg\min \vartheta^* &\overset{(a)}{\Leftrightarrow} 0 \in \arg\min \vartheta^*(\phi) \\ &\overset{(b)}{\Leftrightarrow} \vartheta^*(\phi) + \vartheta^{**}(0) = 0 \\ &\overset{(c)}{\Leftrightarrow} \vartheta^{***}(\phi) + \vartheta^{**}(0) = 0 \\ &\overset{(d)}{\Leftrightarrow} \phi \in \partial\vartheta^{**}(0)\,,\end{aligned}$$

where the equivalence (a) is due to Prop. 1.22, (b) is due to Prop. 2.9, (c) uses the fact that $\vartheta^* = \vartheta^{***}$ which follows from Th. 2.5 and (d) is due to Prop. 2.9 again. This completes the proof of the third point when ϑ is proper. If ϑ is not proper, then ϑ^* is everywhere $+\infty$ or everywhere $-\infty$. In both cases, $\arg\min \vartheta^* = \mathcal{Y}$. □

The quantity $(p - d)$ is called the **duality gap**.

Theorem 3.2. *It holds that $p \geq d$. Moreover, if ϑ is convex and if*

$$0 \in \operatorname{ri}\operatorname{dom}\vartheta\,, \tag{3.1.1}$$

then the following holds:

1. *$d = p < +\infty$.*
2. *The set of dual solutions $\arg\max \mathcal{D}$ is non empty, and coincides with $-\partial\vartheta(0)$.*

Proof. By Prop. 2.4, $p \geq d$. We prove that $p < +\infty$. By Eq. (3.1.1), $0 \in \operatorname{dom}\vartheta$, thus $\vartheta(0) < +\infty$, which reads $p < +\infty$ by Lemma 3.1.
First assume that ϑ is proper. Th. 2.10 along with Eq. (3.1.1) imply that $\vartheta(0) = \vartheta^{**}(0)$ which reads $p = d$ by Lemma 3.1. The first point is proved. This also implies that $\partial\vartheta(0) = \partial\vartheta^{**}(0)$. By Lemma 3.1, we obtain that $\arg\max \mathcal{D} = -\partial\vartheta(0)$ (and this set is non empty). Hence, the second point is proved.
Finally, consider the case where ϑ is not proper. Since, $\vartheta(0) < +\infty$, this means that $-\infty \in \vartheta(\mathcal{Y})$. By contradiction, assume that $p \in \mathbb{R}$, which reads $\vartheta(0) \in \mathbb{R}$ by Lemma 3.1. By Th. 1.17, there exists $\phi \in \partial\vartheta(0)$. Thus, ϑ admits $\vartheta(0) + \langle \phi, .\rangle$ as an affine minorant. This contradicts the fact that $-\infty \in \vartheta(\mathcal{Y})$. We have shown that $p = -\infty$, which also implies that $d = -\infty$: the first point is proved. Since $\vartheta(0) = -\infty$, it holds that $\partial\vartheta(0) = \mathcal{Y}$. Since $\arg\max \mathcal{D} = \mathcal{Y}$ by Lemma 3.1, the second point holds as well. □

Eq. (3.1.1) is sometimes refered to as a **qualification condition**.

Remark 3.1. If F is convex, then ϑ is convex by Prop. 1.11.

3.1.2 Lagrangian

Definition 3.1. The **Lagrangian** associated with the function F is the mapping on $\mathcal{X} \times \mathcal{Y} \to [-\infty, +\infty]$ defined by:

$$\mathcal{L} : (x, \phi) \mapsto \inf_{y \in \mathcal{Y}} \left(F(x, y) + \langle \phi, y\rangle\right)\,.$$

Assumption 3.1. For all $x \in \mathcal{X}$, $F(x, .)$ is convex and l.s.c. at zero.

Proposition 3.3. *For every $\phi \in \mathcal{Y}$,*

$$\mathcal{D}(\phi) = \inf \mathcal{L}(\mathcal{X}, \phi)\,.$$

Moreover, under Assumption 3.1, for every $x \in \mathcal{X}$,

$$\mathcal{P}(x) = \sup \mathcal{L}(x, \mathcal{Y})\,.$$

Proof. After straightforward algebra, we obtain for every $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$,

$$\mathcal{L}(x, \phi) = -\sup_{y \in \mathcal{Y}} \langle -\phi, y \rangle - F(x, y)\,. \tag{3.1.2}$$

Thus, $\inf \mathcal{L}(\mathcal{X}, \phi) = -\sup_{(x,y)} \langle -\phi, y \rangle - F(x, y) = -F^*(0, -\phi) = \mathcal{D}(\phi)$. To prove the second point, set $x \in \mathcal{X}$ and consider the mapping $F_x : y \mapsto F(x, y)$. By Eq. (3.1.2), $\mathcal{L}(x, \phi) = -F_x^*(-\phi)$. Thus, $\sup_\phi \mathcal{L}(x, \phi) = F_x^{**}(0)$. By Th. 2.8, $F_x^{**}(0) = F_x(0)$ and since $\mathcal{P}(x) = F_x(0)$, the result is proved. □

As an immediate consequence of Prop. 3.3, the following equalities hold under Assumption 3.1:

$$p = \inf_{x \in \mathcal{X}} \sup_{\phi \in \mathcal{Y}} \mathcal{L}(x, \phi) \tag{3.1.3}$$

$$d = \sup_{\phi \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \mathcal{L}(x, \phi)\,. \tag{3.1.4}$$

Definition 3.2. We say that (x, ϕ) is a **saddle point** of $\mathcal{L}$ if

$$\inf \mathcal{L}(\mathcal{X}, \phi) = \mathcal{L}(x, \phi) = \sup \mathcal{L}(x, \mathcal{Y})\,.$$

Theorem 3.4. *Under Assumption 3.1, the following properties are equivalent:*

i) x is primal optimal, ϕ is dual optimal and $p = d$.

ii) (x, ϕ) is a saddle point of the Lagrangian.

Proof. *i)* $\Rightarrow$ *ii)* $d = \mathcal{D}(\phi) = \inf \mathcal{L}(\mathcal{X}, \phi) \le \mathcal{L}(x, \phi) \le \sup \mathcal{L}(x, \mathcal{Y}) = \mathcal{P}(x) = p$.
ii) $\Rightarrow$ *i)* As (x, ϕ) is a saddle point, we have on the one hand $\mathcal{L}(x, \phi) = \sup \mathcal{L}(x, \mathcal{Y}) = \mathcal{P}(x)$ and on the other hand $\mathcal{L}(x, \phi) = \inf \mathcal{L}(\mathcal{X}, \phi) = \mathcal{D}(\phi)$. Thus, $\mathcal{P}(x) = \mathcal{D}(\phi)$. Hence $p \le \mathcal{P}(x) = \mathcal{D}(\phi) \le d$. We conclude using $p \ge d$ (Th. 3.2). □

Corollary 3.5. *Assume that ϑ is convex and let Assumption 3.1 hold true. If*

$$0 \in \operatorname{ri} \operatorname{dom} \vartheta\,,$$

then $d = p < +\infty$ and the two following properties are equivalent:

i) x is primal optimal.

ii) There exists $\phi \in \mathcal{Y}$ such that (x, ϕ) is a saddle point of the Lagrangian.

In that case, every ϕ satisfying ii) is a dual solution.

Proof. *i)* $\Rightarrow$ *ii)*. By Th. 3.2, $d = p < +\infty$ and there exists $\phi \in \arg\max \mathcal{D}$. By Th. 3.4, (x, ϕ) is a saddle point of $\mathcal{L}$.
ii) $\Rightarrow$ *i)* is an immediate consequence of Th. 3.4. The same holds for the last statement of the theorem. □

3.2 Fenchel-Rockafellar duality

In this section, we introduce two functions $f : \mathcal{X} \to (-\infty, +\infty]$, $g : \mathcal{Y} \to (-\infty, +\infty]$ and a linear operator $M : \mathcal{X} \to \mathcal{Y}$. We address the special case where $F : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]$ is given by

$$F : (x, y) \mapsto f(x) + g(Mx - y)\,. \tag{3.2.1}$$

Eq. (3.2.1) is a standing assumption throughout Section 3.2.

3.2.1 Main results

Proposition 3.6. *Let $f : \mathcal{X} \to (-\infty, +\infty]$ and $g : \mathcal{Y} \to (-\infty, +\infty]$ be two functions and let $M : \mathcal{X} \to \mathcal{Y}$ be a linear operator. The following holds.*

i) The primal function $\mathcal{P} : \mathcal{X} \to [-\infty, +\infty]$ is given by

$$\mathcal{P} : x \mapsto f(x) + g(Mx)\,.$$

ii) The Lagrangian $\mathcal{L} : \mathcal{X} \times \mathcal{Y} \to [-\infty, +\infty]$ is given by

$$\mathcal{L} \,:\, (x, \phi) \mapsto \begin{cases} f(x) + \langle \phi, Mx \rangle - g^*(\phi) & \text{if } x \in \operatorname{dom} f\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2.2}$$

iii) The dual function $\mathcal{D}$ on $\mathcal{Y} \to [-\infty, +\infty]$ is given by

$$\mathcal{D} : \phi \mapsto -f^*(-M^*\phi) - g^*(\phi)\,.$$

Proof. Point (*i*) is immediate. For every $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$, $\mathcal{L}(x, \phi) = \inf_{y \in \mathcal{Y}} f(x) + g(Mx - y) + \langle \phi, y \rangle$. In particular, $\mathcal{L}(x, \phi) = +\infty$ if $x \notin \operatorname{dom} f$. If $x \in \operatorname{dom} f$,

$$\begin{aligned} \mathcal{L}(x, \phi) &= f(x) + \langle \phi, Mx \rangle + \inf_{y \in \mathcal{Y}} g(Mx - y) - \langle \phi, Mx - y \rangle \\ &= f(x) + \langle \phi, Mx \rangle + \inf_{w \in \mathcal{Y}} g(w) - \langle \phi, w \rangle \\ &= f(x) + \langle \phi, Mx \rangle - \sup_{w \in \mathcal{Y}} \langle \phi, w \rangle - g(w)\,, \end{aligned}$$

which proves (*ii*). By Prop. 3.3, $\mathcal{D}(\phi) = \inf \mathcal{L}(\mathcal{X}, \phi)$ for all $\phi \in \mathcal{Y}$. Hence,

$$\begin{aligned} \mathcal{D}(\phi) &= \inf_{x \in \operatorname{dom} f} (f(x) + \langle \phi, Mx \rangle - g^*(\phi)) \\ &= -\sup_{x \in \operatorname{dom} f} (\langle -\phi, Mx \rangle - f(x)) - g^*(\phi) \\ &= -f^*(-M^*\phi) - g^*(\phi)\,. \end{aligned}$$

This proves (*iii*). □

Theorem 3.7 (Fenchel-Rockafellar)**.** *Let $f : \mathcal{X} \to (-\infty, +\infty]$ and $g : \mathcal{Y} \to (-\infty, +\infty]$ be two convex functions and let $M : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Assume that*

$$0 \in \operatorname{ri}(M \operatorname{dom} f - \operatorname{dom} g)\,. \tag{3.2.3}$$

Then,

$$\inf_{x \in \mathcal{X}} f(x) + g(Mx) = -\min_{\phi \in \mathcal{Y}} f^*(-M^*\phi) + g^*(\phi)\,. \tag{3.2.4}$$

Proof. The value function associated with the function F defined in (3.2.1) is the function $\vartheta : \mathcal{Y} \to [-\infty, +\infty]$ given by $\vartheta : y \mapsto \inf_{x \in \mathcal{X}} f(x) + g(Mx - y)$. As f, g are convex, F is convex by Prop. 1.13 and Prop. 1.14. Thus, ϑ is convex by Prop. 1.12. A point y belongs to $\operatorname{dom} \vartheta$ iff there exists $x \in \mathcal{X}$ such that $f(x) + g(Mx - y) < +\infty$. This is equivalent to the existence of $x \in \operatorname{dom} f$, such that $Mx - y \in \operatorname{dom} g$. Otherwise stated, $y \in Mx - \operatorname{dom} g$. This shows that $\operatorname{dom} \vartheta = M \operatorname{dom} f - \operatorname{dom} g$. Thus condition (3.2.3) reads $0 \in \operatorname{ri} \operatorname{dom} \vartheta$. By Th. 3.2, we obtain that $\inf \mathcal{P} = \max \mathcal{D}$. This is equivalent to (3.2.4) by Prop. 3.6. □

Proposition 3.8. *Consider a function $f : \mathcal{X} \to (-\infty, +\infty]$, a function $g \in \Gamma_0(\mathcal{Y})$ and a linear map $M : \mathcal{X} \to \mathcal{Y}$. Let (x, ϕ) be a point in $\mathcal{X} \times \mathcal{Y}$. The following statements are equivalent:*

i) (x, ϕ) *satisfies*

$$0 \in \partial f(x) + M^* \phi \tag{3.2.5a}$$
$$0 \in -Mx + \partial g^*(\phi)\,. \tag{3.2.5b}$$

ii) (x, ϕ) *is a saddle point of the Lagrangian.*

iii) x *is a primal solution,* ϕ *is a dual solution, and* $p = d$.

Proof. As $g \in \Gamma_0(\mathcal{Y})$, the Assumption 3.1 is satisfied for F given by (3.2.1). By Th. 3.4, this shows that *ii)* $\Leftrightarrow$ *iii)*.
By Prop. 3.6, (x, ϕ) is a saddle point of the Lagrangian (3.2.2) iff

$$\begin{cases} x \in \operatorname{dom} f \\ x \in \arg\min_{w \in \mathcal{X}} \left(f(w) + \langle \phi, Mw \rangle - g^*(\phi)\right) \\ \phi \in \arg\min_{\psi \in \mathcal{Y}} \left(f(x) - \langle \psi, Mx \rangle + g^*(\psi)\right)\,. \end{cases} \tag{3.2.6}$$

By Fermat's rule (Prop. 1.22) and Exercice 1.5, the last two inclusions are equivalent to Eq. (3.2.5). Since ∂f takes the value $\emptyset$ outside $\operatorname{dom} f$, the condition $x \in \operatorname{dom} f$ can be discarded from 3.2.6. Thus, *i)* $\Leftrightarrow$ *ii)*. □

Proposition 3.9. *Consider a convex function $f : \mathcal{X} \to (-\infty, +\infty]$, a function $g \in \Gamma_0(\mathcal{Y})$ and a linear map $M : \mathcal{X} \to \mathcal{Y}$. Assume that condition (3.2.3) holds. Then, a point $x \in \mathcal{X}$ is primal optimal iff there exists $\phi \in \mathcal{Y}$ s.t. the condition (3.2.5) holds (in which case, any such ϕ is dual optimal).*

Proof. As in the proof of Th. 3.7, the stated hypotheses imply that ϑ is convex and $0 \in \operatorname{ri} \operatorname{dom} \vartheta$. The result is an application of Cor. 3.5 along with the characterization of the saddle points of $\mathcal{L}$ by Prop. 3.8. □

3.2.2 Affine equality constraints

Here, we make the hypothesis that $g := \iota_{\{b\}}$ where $b \in \mathcal{Y}$. This will be a standing assumption in Section 3.2.2.
In this case, the primal optimal points coincide with the solutions to the following minimization problem:

Minimization under affine equality constraints.

$$\text{Minimize } f(x) \text{ w.r.t. } x \in \mathcal{X} \text{ s.t. } Mx = b\,. \tag{3.2.7}$$

The set $\{x \in \operatorname{dom} f \,:\, Mx = b\}$ is called the **feasible set**. A point of $\mathcal{X}$ is said **feasible** if it belongs to the feasible set.

Proposition 3.10. *Consider a function $f : \mathcal{X} \to (-\infty, +\infty]$, a linear map $M : \mathcal{X} \to \mathcal{Y}$ and a point $b \in \mathcal{Y}$. Then,* *i) The primal function is*

$$\mathcal{P} : x \mapsto \begin{cases} f(x) & \textit{if } x \textit{ is feasible} \\ +\infty & \textit{otherwise.} \end{cases}$$

The primal optimal points are the solutions to Problem (3.2.7).

ii) The lagrangian $\mathcal{L} : \mathcal{X} \times \mathcal{Y} \to [-\infty, +\infty]$ *is given by*

$$\mathcal{L} : (x, \phi) \mapsto \begin{cases} f(x) + \langle \phi, Mx - b \rangle & \textit{if } x \in \operatorname{dom} f \\ +\infty & \textit{otherwise.} \end{cases}$$

iii) The dual function is $\mathcal{D} : \phi \mapsto -f^*(-M^*\phi) - \langle b, \phi \rangle$.

Proof. Apply Prop. 3.6 with $g := \iota_{\{b\}}$. In this case, $g \circ M = \iota_{M^{-1}(b)}$. This proves point (*i*). Points (*ii*-*iii*) follows from the identity $g^* = \langle \,.\,, b \rangle$. □

Proposition 3.11. *Consider a function* $f : \mathcal{X} \to (-\infty, +\infty]$*, a linear map* $M : \mathcal{X} \to \mathcal{Y}$ *and a point* $b \in \mathcal{Y}$*. Let* (x, ϕ) *be a point in* $\mathcal{X} \times \mathcal{Y}$*. The following statements are equivalent.*

i) (x, ϕ) *satisfies*

$$\begin{cases} x \textit{ is feasible} \\ 0 \in \partial f(x) + M^*\phi \,. \end{cases} \tag{3.2.8}$$

ii) (x, ϕ) *is a saddle point of the Lagrangian.*

iii) x *is a primal solution,* ϕ *is a dual solution, and* $p = d$*.*

Proof. Use Prop. 3.8 with $g = \iota_{\{b\}}$. □

Proposition 3.12. *Consider a convex function* $f : \mathcal{X} \to (-\infty, +\infty]$*, a linear map* $M : \mathcal{X} \to \mathcal{Y}$ *and a point* $b \in \mathcal{Y}$*. Assume that*

$$\exists x_0 \in \operatorname{ri}(\operatorname{dom} f),\ Mx_0 = b \,. \tag{3.2.9}$$

Then, a point $x \in \mathcal{X}$ *is primal optimal iff there exists* $\phi \in \mathcal{Y}$ *s.t. the conditions (3.2.8) hold (in which case, any such* ϕ *is dual optimal).*

Proof. Set $g = \iota_{\{b\}}$. Condition (3.2.3) is equivalent to $b \in \operatorname{ri}(M \operatorname{dom} f)$. By Th. 1.7, this is again equivalent to $b \in M \operatorname{ri} \operatorname{dom} f$, hence the condition (3.2.9). The result follows from Prop. 3.9. □

3.2.3 Operations on subdifferentials

Proposition 3.13. *Let* $f : \mathcal{X} \to (-\infty, +\infty]$ *and* $g : \mathcal{Y} \to (-\infty, +\infty]$ *be two convex functions and* $M : \mathcal{X} \to \mathcal{Y}$ *be a linear operator. Consider* $x \in \mathcal{X}$*. Then,*

$$\partial f(x) + M^* \partial g(Mx) \subset \partial (f + g \circ M)(x) \,. \tag{3.2.10}$$

Moreover, if the following condition holds:

$$0 \in \operatorname{ri}(M \operatorname{dom} f - \operatorname{dom} g) \,,$$

then

$$\partial (f + g \circ M)(x) = \partial f(x) + M^* \partial g(Mx) \,.$$

Proof. In order to prove (3.2.10), consider $\phi \in \partial f(x)$ and $\psi \in \partial g(Mx)$. For every $y \in \mathcal{X}$, it holds that

$$f(y) \geq f(x) + \langle \phi, y - x \rangle$$
$$g(My) \geq g(Mx) + \langle \psi, M(y - x) \rangle \ .$$

Summing the inequalities, we obtain $(f + g \circ M)(y) \geq (f + g \circ M)(x) + \langle \phi + M^* \psi, y - x \rangle$. This proves that $\phi + M^*\psi \in \partial(f + g \circ M)(x)$ and Eq. (3.2.10) follows.
Let $x \in \mathcal{X}$ and $\phi \in \mathcal{X}$. Applying Theorem 3.7 on $(f - \langle \phi, . \rangle)$ and g, we can see that there exists $\psi \in \mathcal{Y}$ such that $f(x) - \langle \phi, x \rangle + g(Mx) = -(f - \langle \phi, . \rangle)^*(-M^*\psi) - g^*(\psi)$. One can easily check that $(f - \langle \phi, . \rangle)^* = f^*(. + \phi)$. Hence, $f(x) - \langle \phi, x \rangle + g(Mx) = -f(-M^*\psi + \phi) - g^*(\psi)$. Said otherwise,

$$f(x) + f(-M^*\psi + \phi) - \langle \phi, x \rangle + g(Mx) + g^*(\psi) = 0\,,$$

and so

$$[f(x) + f(-M^*\psi + \phi) - \langle -M^*\psi + \phi, x \rangle] + [g(Mx) + g^*(\psi) - \langle \psi, Mx \rangle] = 0\,.$$

Each of the two terms in the brackets is nonnegative according to Fenchel-Young inequality. Thus both are null. The equality case in Fenchel-Young inequality allows us to conclude that $\psi \in \partial g(Mx)$ et $-M^*\psi + \phi \in \partial f(x)$. The last inclusion can be read also as $\phi \in \partial f(x) + M^*\partial g(Mx)$ which is . ☐

Corollary 3.14. *Let $m \geq 2$ be an integer and consider functions $f_1, \cdots, f_m : \mathcal{X} \to (-\infty, +\infty]$. Then, $\partial f_1 + \cdots + \partial f_m \subset \partial\,(f_1 + \cdots + f_m)$.*
Moreover, if the functions are convex and satisfy the condition

$$0 \in \bigcap_{i=2}^{m} \mathrm{ri} \left(\mathrm{dom}\, f_i - \bigcap_{j=1}^{i-1} \mathrm{dom}\, f_j \right)$$

or the stronger condition

$$0 \in \bigcap_{i=1}^{m} \mathrm{ri}\,\mathrm{dom}\, f_i \ .$$

Then, $\partial\,(f_1 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$.

Proof. By induction using Prop. 3.13. ☐

3.2.4 The Attouch-Brezis theorem*

Let f and g be two functions on $\mathcal{X} \to (-\infty, +\infty]$. We recall that $(f \,\square\, g)^* = f^* + g^*$ (see Exercise 2.6).

Theorem 3.15 (Attouch-Brezis). *Let $f, g : \mathcal{X} \to (-\infty, +\infty]$ be two convex functions such that*

$$0 \in \mathrm{ri}(\mathrm{dom}\, f - \mathrm{dom}\, g)\,. \tag{3.2.11}$$

Then $(f + g)^* = f^* \,\square\, g^*$, and this function belongs to $\Gamma_0(\mathcal{X})$. Moreover, the infimal convolution is exact.

Proof. Let $\psi \in \mathcal{X}$. Note that $(f + g)^*(\psi) = -\inf_{x \in \mathcal{X}} (f - \langle \psi, . \rangle)(x) + g(x)$. Clearly, $\mathrm{dom}(f - \langle \psi, . \rangle) = \mathrm{dom}\, f$. Thus, Eq. (3.2.11) and Th. 3.7 imply that $(f + g)^*(\psi) = \min_{\phi \in \mathcal{X}} (f - \langle \psi, . \rangle)^*(-\phi) + g^*(\phi)$. It is straightforward to show that $(f - \langle \psi, . \rangle)^*(-\phi) = f^*(\psi - \phi)$, which proves the first part of the result.
Finally, the condition (3.2.11) implies that $0 \in \mathrm{dom}\, f - \mathrm{dom}\, g$ which also reads $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. By Remark 2.1, $(f + g)^*$ does not take the value $-\infty$. Also Prop. 2.3 and Prop. 1.18, $(f + g)^*$ is not identically $+\infty$. Hence, $(f + g)^*$ is proper. ☐

3.3 Lagrangian duality

In this paragraph, we consider an integer $m \geq 1$. For every $(a, b) \in \mathbb{R}^m \times \mathbb{R}^m$, the notations $a \leq b$ (or $b \geq a$) is used as an equivalent to $a - b \in (-\infty, 0]^m$. Similarly, we write $a < b$ as a short for $a - b \in (-\infty, 0)^m$. Finally, we denote by $a_1, \dots, a_m$ the real components of any vector $a \in \mathbb{R}^m$.

3.3.1 Inequality and affine equality constraints

Consider a function $f : \mathcal{X} \to (-\infty, +\infty]$, a linear operator $M : \mathcal{X} \to \mathcal{Y}$, a point $b \in \mathcal{Y}$, and a map $g : \mathcal{X} \to \mathbb{R}^m$. We address the case where the function $F : \mathcal{X} \times (\mathcal{Y} \times \mathbb{R}^m) \to [-\infty, +\infty]$ is defined for every $x \in \mathcal{X}$, $u \in \mathcal{Y}$, $v \in \mathbb{R}^m$, by

$$F(x, (u, v)) = f(x) + \iota_{\{b\}}(Mx - u) + \iota_{(-\infty,0]^m}(g(x) - v) \,. \tag{3.3.1}$$

We consider Eq. (3.3.1) as a standing assumption throughout Section 3.3.
In this case, the primal optimal points coincide with the solutions to the following minimization problem:

Minimization under inequality and affine equality constraints.

$$\text{minimize } f(x) \text{ w.r.t. } x \in \mathcal{X} \text{ s.t. } Mx = b \text{ and } g(x) \leq 0 \,. \tag{3.3.2}$$

The set $\{x \in \operatorname{dom} f \;:\; Mx = b,\, g(x) \leq 0\}$ is called the **feasible set**. A point of $\mathcal{X}$ is said **feasible** if it belongs to the feasible set.

Proposition 3.16. *Consider a function $f : \mathcal{X} \to (-\infty, +\infty]$, a linear map $M : \mathcal{X} \to \mathcal{Y}$, a point $b \in \mathcal{Y}$ and a map $g : \mathcal{X} \to \mathbb{R}^m$. Then,*

i) The primal function is

$$\mathcal{P} : x \mapsto \begin{cases} f(x) & \text{if } x \text{ is feasible} \\ +\infty & \text{otherwise.} \end{cases}$$

The primal optimal points are the solutions to Problem (3.3.2).

ii) The lagrangian $\mathcal{L} : \mathcal{X} \times (\mathcal{Y} \times \mathbb{R}^m) \to [-\infty, +\infty]$ is defined by

$$\mathcal{L} : (x, (\lambda, \nu)) \mapsto \begin{cases} f(x) + \langle \lambda, Mx - b \rangle + \langle \nu, g(x) \rangle - \iota_{[0,+\infty)^m}(\nu) & \text{if } x \in \operatorname{dom} f \\ +\infty & \text{otherwise.} \end{cases} \tag{3.3.3}$$

Proof. The first point is immediate. We prove the second. Recall that $\mathcal{L}(x, (\lambda, \nu)) = \inf_{u,v} F(x, (u, v)) + \langle \lambda, u \rangle + \langle \nu, v \rangle$. By the definition of F in Eq. (3.3.1), $\mathcal{L}(x, (\lambda, \nu)) = +\infty$ when $x \notin \operatorname{dom} f$. If $x \in \operatorname{dom} f$,

$$\begin{aligned}
\mathcal{L}(x, (\lambda, \nu)) &= f(x) + \inf_{u \in \mathcal{Y}} \big(\iota_{\{b\}}(Mx - u) + \langle \lambda, u \rangle\big) + \inf_{v \in \mathbb{R}^m} \big(\iota_{(-\infty,0]^m}(g(x) - v) + \langle \nu, v \rangle\big) \\
&= f(x) + \langle \lambda, Mx - b \rangle + \langle \nu, g(x) \rangle \\
&\qquad + \inf_{u \in \mathcal{Y}} \big(\iota_{\{0\}}(Mx - b - u) - \langle \lambda, Mx - b - u \rangle\big) \\
&\qquad + \inf_{v \in \mathbb{R}^m} \big(\iota_{(-\infty,0]^m}(g(x) - v) - \langle \nu, g(x) - v \rangle\big) \\
&= f(x) + \langle \lambda, Mx - b \rangle + \langle \nu, g(x) \rangle + \inf_{w \in \mathbb{R}^m} \big(\iota_{(-\infty,0]^m}(w) - \langle \nu, w \rangle\big) \,.
\end{aligned}$$

where we use the change of variables $w = g(x) - v$ in the last equation. The proof of point (*ii*) is completed upon noting that $-\inf_{w\in\mathbb{R}^m}\big(\iota_{(-\infty,0]^m}(w) - \langle \nu, w\rangle\big) = \iota^*_{(-\infty,0]^m}(\nu) = \iota_{[0,+\infty)^m}(\nu)$ (see Exercise 2.2). □

We denote by $g_1, \cdots, g_m : \mathcal{X} \to \mathbb{R}$ the components of g *i.e.*, $g : x \mapsto (g_1(x), \cdots, g_m(x))$.

Theorem 3.17. *Consider a proper function $f : \mathcal{X} \to (-\infty, +\infty]$, a linear map $M : \mathcal{X} \to \mathcal{Y}$, a point $b \in \mathcal{Y}$ and a map $g : \mathcal{X} \to \mathbb{R}^m$. Let (x, λ, ν) be a point in $\mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m$. Consider the following statements.*

i) (x, λ, ν) *satisfies*

$$\begin{cases} x \text{ is feasible} & \text{(3.3.4a)}\\ \nu \geq 0 & \text{(3.3.4b)}\\ 0 \in \partial f(x) + M^*\lambda + \sum_{i=1}^m \nu_i\, \partial g_i(x) & \text{(3.3.4c)}\\ \forall i \in \{1, \dots, m\},\ \nu_i g_i(x) = 0\,. & \text{(3.3.4d)} \end{cases}$$

ii) $(x, (\lambda, \nu))$ *is a saddle point of the Lagrangian.*

iii) x *is a primal solution,* (λ, ν) *is a dual solution, and* $p = d$.

Then, i) ⇒ ii) ⇔ iii). When f, $g_1, \cdots, g_m$ are convex, the three points are equivalent.

Proof. Note that Assumption 3.1 holds when F is defined as in (3.3.1). By Th. 3.4, *ii)* ⇔ *iii)*. We now prove that *i)* ⇒ *ii)*. As f is proper, a point $(x, (\lambda, \nu))$ can be a saddle point of $\mathcal{L}$ only if $x \in \operatorname{dom} f$ and $\nu \geq 0$. Therefore, $(x, (\lambda, \nu))$ is a saddle point of $\mathcal{L}$ iff the following four conditions hold together:

$$\begin{aligned} & x \in \operatorname{dom} f \\ & \nu \geq 0 \\ & x \in \arg\min_{x'\in\mathcal{X}} f(x') + \langle \lambda, Mx'\rangle + \langle \nu, g(x')\rangle \\ & (\lambda, \nu) \in \arg\max_{(\lambda',\nu')\in\mathcal{Y}\times\mathbb{R}^m} \langle \lambda, Mx - b\rangle + \langle \nu', g(x)\rangle - \iota_{[0,+\infty)^m}(\nu')\,. \end{aligned} \tag{3.3.5}$$

Using Exercise 1.3, Eq. (3.3.5) is equivalent to

$$\begin{cases} \lambda \in \arg\min_{\lambda'\in\mathcal{Y}} \langle \lambda', -Mx + b\rangle \\ \nu_i \in \arg\min_{\nu'\in\mathbb{R}} \iota_{[0,+\infty)}(\nu') - \nu'\, g_i(x) \quad (\forall i \in \{1, \cdots, m\})\,, \end{cases}$$

By Fermat's rule (Prop. 1.22) and the fact Exercise 1.6, the above condition is equivalent to

$$\begin{cases} 0 = -Mx + b \\ 0 \in N_{[0,+\infty)}(\nu_i) - g_i(x) \quad (\forall i \in \{1, \cdots, m\})\,, \end{cases}$$

where we recall that $N_{[0,+\infty)}(a)$ is the normal cone of $[0, +\infty)$, equal to $\{0\}$ is $a > 0$, to $(-\infty, 0]$ is $a = 0$, and to the empty set if $a < 0$. Hence, for a given $i \in \{1, \cdots, m\}$, condition $g_i(x) \in N_{[0,+\infty)}(\nu_i)$ reduces to $\nu_i \geq 0$, $g_i(x) \leq 0$ and $\nu_i\, g_i(x) = 0$. Putting all pieces together, $(x, (\lambda, \nu))$ is a saddle point of $\mathcal{L}$ iff

$$\begin{aligned} & x \in \operatorname{dom} f,\ Mx = b,\ g(x) \leq 0 \\ & \nu \geq 0 \\ & 0 \in \partial\left(f + \langle \lambda, M\,.\rangle + \langle \nu, g(\,.\,)\rangle\right)(x) \\ & \nu_i\, g_i(x) = 0 \quad (\forall i \in \{1, \cdots, m\})\,. \end{aligned} \tag{3.3.6}$$

where we applied Fermat's rule (Prop. 1.22) to obtain the condition (3.3.6). Using Cor. 3.14, it holds that

$$\partial f(x) + M^*\lambda + \sum_{i=1}^{m} \nu_i \partial g_i(x) \subset \partial\left(f + \langle \lambda, M\,.\,\rangle + \langle \nu, g(\,.\,)\rangle\right)(x)\,. \tag{3.3.7}$$

Thus, the condition $0 \in \partial f(x) + M^*\lambda + \sum_{i=1}^m \nu_i \partial g_i(x)$ implies Eq. (3.3.6). Thus, *i)* $\Rightarrow$ *ii)*. As f is proper and $\operatorname{dom} g_i = \mathcal{X}$ for all $i \in \{1, \cdots, m\}$, the inclusion (3.3.7) holds with equality if f, $g_1, \ldots, g_m$ are convex (use again Cor. 3.14). In the latter case, *i)* $\Leftrightarrow$ *ii)*. □

The set of conditions in Eq. (3.3.4) are often refered to as the **Karush-Kuhn-Tucker (KKT) conditions**. Eq. (3.3.4a) is often refered to as the primal feasibility condition. Eq. (3.3.4b) is often refered to as the dual feasibility condition, because it is seen from the expression of the Lagrangian in (3.3.3) that the dual function $-\inf \mathcal{L}(\mathcal{X}, \,.\,)$ is greater than $-\infty$ only if $\nu \geq 0$. Finally, the m conditions in Eq. (3.3.4d) are called the **complementary slackness conditions**.

Theorem 3.18. *Consider a proper function $f : \mathcal{X} \to (-\infty, +\infty]$, a linear map $M : \mathcal{X} \to \mathcal{Y}$, a point $b \in \mathcal{Y}$ and a map $g : \mathcal{X} \to \mathbb{R}^m$. Assume that f, g_1, ..., g_m are convex and that*

$$\exists x_0 \in \operatorname{ri}(\operatorname{dom} f),\ Mx_0 = b \text{ and } g(x_0) < 0\,. \tag{3.3.8}$$

Then, a point $x \in \mathcal{X}$ is primal optimal iff there exists $(\lambda, \nu) \in \mathcal{Y} \times \mathbb{R}^m$ s.t. the conditions (3.3.4) hold (in which case, any such (λ, ν) is dual optimal).

Proof. The result is an application of Cor. 3.5 along with the characterization of the saddle points of $\mathcal{L}$ by Th. 3.17. It is thus sufficient to check the hypotheses of Cor. 3.5 when F is defined as in (3.3.1). Assumption 3.1 clearly holds. The value function ϑ is defined on $\mathcal{Y} \times \mathbb{R}^m \to [-\infty, +\infty]$ by $\vartheta : (u, v) \mapsto f(x) + \iota_{\{b\}}(Mx - u) + \iota_{(-\infty,0]^m}(g(x) - v)$. As f, $g_1, \ldots, g_m$ are convex, so is ϑ. We must check that $0 \in \operatorname{ri}\operatorname{dom}\vartheta$ and the proof is completed. A point (u, v) is $\operatorname{dom}\vartheta$ iff there exists $x \in \operatorname{dom} f$ s.t. $u = Mx - b$ and $v \geq g(x)$. Define $C := \{(x, v) : x \in \operatorname{dom} f, v \in \mathbb{R}^m, v \geq g(x)\}$. It holds that

$$\operatorname{dom}\vartheta = \mathcal{M}\,C - (b, 0)\,,$$

where $\mathcal{M} : \mathcal{X} \times \mathbb{R}^m \to \mathcal{Y} \times \mathbb{R}^m$ is the linear operator defined by $\mathcal{M} : (x, v) \mapsto (Mx, v)$. Note that C is convex. By Th. 1.7,

$$\operatorname{ri}\operatorname{dom}\vartheta = \mathcal{M}\ \operatorname{ri}(C) - (b, 0)\,.$$

We shall prove below that $(x_0, 0) \in \operatorname{ri}(C)$. As $\mathcal{M}\,.(x_0, 0) - (b, 0) = (0, 0)$, this implies that $(0, 0) \in \operatorname{ri}\operatorname{dom}\vartheta$, and concludes the proof.
We now show that $(x_0, 0) \in \operatorname{ri}(C)$. As $x_0 \in \operatorname{ri}\operatorname{dom} f$, there exists $\varepsilon > 0$ s.t. $B_{\mathcal{X}}(x_0, \varepsilon) \cap \operatorname{aff}(\operatorname{dom} f) \subset \operatorname{dom} f$ where $B_{\mathcal{X}}(x_0, \varepsilon)$ represents the open Euclidean ball of center x_0 and radius ε. Moreover, for every $i \in \{1, \cdots, m\}$, $g_i(x_0) < 0$. Note that g_i is continuous by Prop. 1.10. Thus, there exists $\varepsilon' > 0$ s.t. for every i, $\sup\{g_i(x) : x \in B_{\mathcal{X}}(x_0, \varepsilon')\} < g_i(x_0)/2$. Define

$$V := B_{\mathcal{X}}(x_0, \varepsilon \wedge \varepsilon') \times \prod_{i=1}^{m} (g_i(x_0)/2, -g_i(x_0)/2)\,.$$

The set V is a neighborhood of $(x_0, 0)$. We prove that $V \cap \operatorname{aff}(C) \subset C$, which implies that $(x_0, 0) \in \operatorname{ri}(C)$. Note that C is a subset of the affine space $\operatorname{aff}(\operatorname{dom} f) \times \mathbb{R}^m$. Therefore, $\operatorname{aff}(C) \subset \operatorname{aff}(\operatorname{dom} f) \times \mathbb{R}^m$. Thus,

$$V \cap \operatorname{aff}(C) \subset \left(B_{\mathcal{X}}(x_0, \varepsilon \wedge \varepsilon') \cap \operatorname{aff}(\operatorname{dom} f)\right) \times \prod_{i=1}^{m} (g_i(x_0)/2, -g_i(x_0)/2)$$

Let $(x, v) \in V \cap \operatorname{aff}(C)$. It holds that $x \in B_{\mathcal{X}}(x_0, \varepsilon) \cap \operatorname{aff}(\operatorname{dom} f)$, thus $x \in \operatorname{dom} f$. Moreover, for every $i \in \{1, \cdots, m\}$, $v_i > g_i(x_0)/2$. Thus,

$$\nu_i > \sup\{g_i(y) : y \in B_{\mathcal{X}}(x_0, \varepsilon')\}\,,$$

which implies that $v_i > g_i(x)$, because $x \in B_{\mathcal{X}}(x_0, \varepsilon')$. We have shown that $v \geq g(x)$ and that $x \in \operatorname{dom} f$. Stated otherwise, we have shown that $V \cap \operatorname{aff}(C) \subset C$. As a conclusion, $(x_0, 0) \in \operatorname{ri}(C)$. □

Eq. (3.3.8) is refered to as **Slater's condition**.

3.3.2 Inequality constraints only

In this paragraph, we re-state the results of Section 3.3.1 in the special case where only inequality constraints are present. This is obtained by setting $\mathcal{Y} = \{0\}$, $M = 0$ and $b = 0$. The primal problem (3.3.2) reduces to the following problem:

Minimization under inequality constraints.

$$\text{Minimize } f(x) \text{ w.r.t. } x \in \mathcal{X} \text{ s.t. } g(x) \leq 0\,. \tag{3.3.9}$$

The primal optimal points are the solutions to Problem (3.3.9). The Lagrangian $\mathcal{L} : \mathcal{X} \times \mathbb{R}^m \to [-\infty, +\infty]$ is

$$\mathcal{L} : (x, \nu) \mapsto \begin{cases} f(x) + \langle \nu, g(x) \rangle - \iota_{[0,+\infty)^m}(\nu) & \text{if } x \in \operatorname{dom} f \\ +\infty & \text{otherwise.} \end{cases}$$

A point (x, ν) is a saddle point of $\mathcal{L}$ iff it satisfies the KKT conditions (3.3.4) only replacing the third condition (3.3.4c) by the simplest condition

$$0 \in \partial f(x) + \sum_{i=1}^{m} \nu_i \, \partial g_i(x)\,, \tag{3.3.10}$$

and the statement of Th. 3.17 holds under this substitution. Although Th. 3.18 continue to hold, its statement can be simplified in this case. We re-state it as follows.

Theorem 3.19. *Consider a proper function $f : \mathcal{X} \to (-\infty, +\infty]$ and a map $g : \mathcal{X} \to \mathbb{R}^m$. Assume that f, g_1, …, g_m are convex and that*

$$\exists x_0 \in \operatorname{dom} f,\; g(x_0) < 0\,. \tag{3.3.11}$$

Then, a point $x \in \mathcal{X}$ is primal optimal iff there exists $\nu \in \mathbb{R}^m$ s.t. the conditions (3.3.4a), (3.3.4b), (3.3.4d) and (3.3.10) hold (in which case, any such ν is dual optimal).

Proof. As in the proof of Th. 3.18, the point is to establish that Eq. (3.3.11) implies that $0 \in \operatorname{ri} \operatorname{dom} \vartheta$, where ϑ is the value function. When $\mathcal{Y} = \{0\}$, $M = 0$ and $b = 0$, the latter reduces to the function $\vartheta : \mathbb{R}^m \to [-\infty, +\infty]$ s.t. $\vartheta : v \mapsto f(x) + \iota_{(-\infty,0]^m}(g(x) - v)$ and its domain is $\operatorname{dom} \vartheta = \{v : \exists x \in \operatorname{dom} f, v \geq g(x)\}$. Choose ε such that $g_i(x_0) < -\varepsilon$ for all i, where x_0 satisfies Condition (3.3.11). For all $v \in \mathbb{R}^m$ in the ball $B(0, \varepsilon)$ of radius ε, it holds that $v \geq g(x_0)$. Hence, $B(0, \varepsilon) \subset \operatorname{dom} \vartheta$. This implies that $0 \in \operatorname{int} \operatorname{dom} \vartheta$. The proof is concluded upon noting that $\operatorname{int} \operatorname{dom} \vartheta \subset \operatorname{ri} \operatorname{dom} \vartheta$. □

Chapter 4

Fixed Points Algorithms

We wish to obtain numerically a minimizer of a convex function $f : \mathcal{X} \to (-\infty, +\infty]$. By Fermat's rule, this amounts to finding a point x such that $0 \in \partial f(x)$. In the case where f is differentiable, this is equivalent to finding a point x such that $\nabla f(x) = 0$. In this case, a celebrated algorithm is the gradient algorithm, which consists in generating a sequence $(x^k : k = 0, 1, \dots)$ that is defined recursively by the equation

$$x^{k+1} = x^k - \gamma \nabla f(x^k),$$

where γ is a positive step size. Formally, we can write this algorithm as $x^{k+1} = T(x^k)$, where $T(x) = x - \gamma \nabla f(x)$. The minimizers of f coincide with the fixed points of T.
More generally, many optimization algorithms take the form $x^{k+1} = T(x^k)$, where the mapping T is chosen in such a way that its fixed points are solutions of the problem. We thus need to devise conditions on T that guarantee the convergence of the algorithm towards a fixed point.

4.1 α-averaged operators

In all this chapter, an **operator** is a mapping from a nonempty subset D of the Euclidean space $\mathcal{X}$ to $\mathcal{X}$. The image of x by an operator T will be denoted arbitrarily as $T(x)$ or Tx. The composition $T \circ R$ of the operator T with the operator R will be denoted compactly as TR when defined. The set of fixed points of T will be denoted $\mathrm{Fix}(T)$. The identity operator on D is denoted as I.

Definition 4.1. Given a real number $L \geq 0$, the operator R defined on D is said L-Lipschitz if for all $x, y \in D$, $\|Rx - Ry\| \leq L\|x - y\|$.

If $L < 1$, we say that R is a **contraction**. If $L = 1$, R is said **non-expansive**.

Remark 4.1. Banach's fixed point theorem states that a contraction R defined on $\mathcal{X}$ has a unique fixed point, and that each sequence that is defined by the recursion $x^{k+1} = R(x^k)$ converges to this fixed point as $k \to \infty$. However, the contraction is a strong assumption, and Banach's fixed point theorem is often unapplicable in the field of optimization. Regarding the non expansiveness, this assumption is not sufficient, as the counter-example $R = -I$ shows.

Definition 4.2. Let $\alpha \in (0, 1]$. The operator T on D is said α–**averaged** if there exists a non-expansive operator R on D such that $T = \alpha R + (1 - \alpha)I$.

A 1/2-averaged operator is said **firmly non-expansive**.

Proposition 4.1. *Let $\alpha \in (0, 1]$. The following statements are equivalent.*

(i) T is α-averaged.

(ii) For all $x, y \in D$, $\|Tx - Ty\|^2 \le \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x - (I - T)y\|^2$.

Proof. Write $T = \alpha R + (1 - \alpha)I$ where R is non-expansive. Equivalently, $R = I - \frac{1}{\alpha}(I - T)$. Writing $\lambda = \frac{1}{\alpha}$ and $Q = I - T$, we get $R = I - \lambda Q$. We now develop:

$$\|Rx - Ry\|^2 = \lambda^2\|Qx - Qy\|^2 + \|x - y\|^2 - 2\lambda \langle Qx - Qy, x - y\rangle \ .$$

Since R is non-expansive, $\|x - y\|^2 \ge \|Rx - Ry\|^2$, thus,

$$\begin{aligned} 0 &\ge \lambda\|Qx - Qy\|^2 - 2 \langle Qx - Qy, x - y\rangle \\ &= \lambda\|Qx - Qy\|^2 - 2\|x - y\|^2 + 2 \langle Tx - Ty, x - y\rangle \ . \end{aligned}$$

We also have

$$\|Qx - Qy\|^2 = \|x - y\|^2 + \|Tx - Ty\|^2 - 2 \langle Tx - Ty, x - y\rangle \ .$$

By substituting the scalar product in the previous expression, we get

$$0 \ge (\lambda - 1)\|Qx - Qy\|^2 - \|x - y\|^2 + \|Tx - Ty\|^2$$

which is the required inequality. □

Proposition 4.2. *An operator T is firmly non-expansive* iff $\langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2$ *for all $x, y \in D$.*

Proof. Write $\|(I-T)x-(I-T)y\|^2 = \|x-y-(T(x)-T(y))\|^2 = \|x-y\|^2 + \|T(x)-T(y)\|^2 - 2\langle T(x) - T(y), x - y\rangle$ and use Proposition 4.1–(ii). with $\alpha = 1/2$. □

Theorem 4.3 (Krasnosel'skii Mann)**.** *Let $D \subset \mathcal{X}$ be a closed set. Let $0 < \alpha < 1$, and let $T : D \to D$ be an α-averaged operator such that* $\mathrm{Fix}(T) \neq \emptyset$*. Then, each sequence (x^k) verifying the recursion $x^{k+1} = T(x^k)$ with $x^0 \in D$ converges to a point of* $\mathrm{Fix}(T)$*.*

Proof. Let $x^\star \in \mathrm{Fix}(T)$. Since T is α-averaged, Proposition 4.1–(ii). implies that for each k,

$$\begin{aligned} \|x^{k+1} - x^\star\|^2 &= \|Tx^k - Tx^\star\|^2 \\ &\le \|x^k - x^\star\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x^k\|^2 \,, \end{aligned} \tag{4.1.1}$$

where we used the fact that $(I - T)x^\star = 0$. By iterating this inequality, we get that

$$0 \le \|x^{k+1} - x^\star\|^2 \le \|x^0 - x^\star\|^2 - \frac{1-\alpha}{\alpha}\sum_{i=0}^{k}\|(I - T)x^i\|^2$$

which implies that the series $\sum_i \|(I - T)x^i\|^2$ is convergent, thus, $(I - T)x^k \to 0$ as $k \to \infty$. Since the sequence of iterates (x^k) belongs to D, each accumulation point $\bar{x}$ of this sequence belongs to D, and since T is α-averaged, it is continuous, thus, $(I - T)\bar{x} = 0$, in other words, $\bar{x} \in \mathrm{Fix}(T)$.
The inequality (4.1.1) implies moreover that the sequence $(\|x^k - x^\star\|^2)$ is decreasing. In particular, the sequence (x^k) is bounded. It admits an accumulation point that is an element of $\mathrm{Fix}(T)$ as we just showed. As a consequence, the inequality (4.1.1) remains true after replacing the fixed point $x^\star$ (which was arbitrarily chosen) with $\bar{x}$. Therefore, the sequence $(\|x^k - \bar{x}\|^2)$ is decreasing and has zero as an accumulation point. Thus, it converges to zero. We showed that $x^k \to \bar{x}$, where $\bar{x} \in \mathrm{Fix}(T)$. □

Lemma 4.4 (Composition)**.** *Let T and S be two operators from D to D such that T is α-averaged and S is β-averaged, where $0 < \alpha, \beta < 1$. Then, there exists $0 < \delta < 1$ such that the composed operator TS is δ-averaged.*

Proof. For all $x, y \in D$,

$$\begin{aligned}\|(I-TS)x-(I-TS)y\|^2 &= \|(I-S)x-(I-S)y+Sx-Sy-(TSx-TSy)\|^2\\ &= \|(I-S)x-(I-S)y+(I-T)Sx-(I-T)Sy\|^2\\ &\le 2\left(\|(I-S)x-(I-S)y\|^2+\|(I-T)Sx-(I-T)Sy\|^2\right)\end{aligned}$$

where we used the well known inequality $\|a+b\|^2 \le 2(\|a\|^2+\|b\|^2)$. By Proposition 4.1, $\|(I-S)x-(I-S)y\|^2 \le \frac{\beta}{1-\beta}\left(\|x-y\|^2-\|Sx-Sy\|^2\right)$, and similarly for T. Setting $\kappa = \max(\beta/(1-\beta), \alpha/(1-\alpha))$, we have

$$\|(I-TS)x-(I-TS)y\|^2 \le 2\kappa(\|x-y\|^2-\|Sx-Sy\|^2+\|Sx-Sy\|^2-\|TSx-TSy\|^2)\,,$$

and finally, $\|(I-TS)x-(I-TS)y\|^2 \le 2\kappa(\|x-y\|^2-\|TSx-TSy\|^2)$. Setting $\delta = (1+(2\kappa)^{-1})^{-1}$, we get that $2\kappa = \delta/(1-\delta)$, and TS is δ-averaged. □

4.2 The gradient algorithm

Let $f : \mathcal{X} \to (-\infty, +\infty]$. We make the following assumption:

Assumption 4.1. f is convex and differentiable on $\mathcal{X}$, and ∇f is L-Lipschitz.

Such functions are called smooth functions in the field of optimization theory.

Theorem 4.5 (Baillon-Haddad)**.** *Under Assumption 4.1, $\forall x, y \in \mathcal{X}$,*

$$\langle \nabla f(x) - \nabla f(y), x-y\rangle \ge \frac{1}{L}\|\nabla f(x)-\nabla f(y)\|^2\,.$$

In other words, $L^{-1}\nabla f$ is firmly non-expansive.

Proof. We first establish the following inequality. For all x, y,

$$f(y) \le f(x) + \langle \nabla f(x), y-x\rangle + \frac{L}{2}\|y-x\|^2\,. \tag{4.2.1}$$

To that end, we define the function $\varphi(t) = f(x+t(y-x))$ on $\mathbb{R}$, and we observe that $f(x) = \varphi(0)$ and $f(y) = \varphi(1)$. The function φ is differentiable with $\varphi'(t) = \langle \nabla f(x+t(y-x)), y-x\rangle$. Consequently, $f(y) = f(x) + \int_0^1 \langle \nabla f(x+t(y-x)), y-x\rangle\, \mathrm{d}t$. Thus, $f(y) = f(x) + \langle \nabla f(x), y-x\rangle + \delta$, where $\delta = \int_0^1 \langle \nabla f(x+t(y-x)) - \nabla f(x), y-x\rangle\, \mathrm{d}t$. Using the fact that ∇f is L-Lipschitz, Inequality (4.2.1) is straightforward.

Second, we show that

$$f(y) \ge f(x) + \langle \nabla f(x), y-x\rangle + \frac{1}{2L}\|\nabla f(y)-\nabla f(x)\|^2\,. \tag{4.2.2}$$

For this, we fix x and we set $\psi(y) = f(y) - \langle \nabla f(x), y-x\rangle$. We easily verify that ψ is convex, that its derivative $\nabla\psi$ is L-Lipschitz, and that $\nabla\psi(x) = 0$, in other words, x is a minimizer for ψ. In particular, $f(x) = \psi(x) \le \psi(y - \frac{1}{L}\nabla\psi(y))$. We now use the inequality (4.2.1) at the points $y - \frac{1}{L}\nabla\psi(y)$ and y after replacing f with ψ. We get

$$f(x) \le \psi(y) - \frac{1}{L}\langle \nabla\psi(y), \nabla\psi(y)\rangle + \frac{1}{2L}\|\nabla\psi(y)\|^2\,.$$

Inequality (4.2.2) is established by noticing that $\nabla\psi(y) = \nabla f(y) - \nabla f(x)$. The proof of the theorem is completed by adding (4.2.2) to the inequality that we obtain by exchanging x and y in (4.2.2). The last statement of the theorem is a consequence of Proposition 4.2. □

Lemma 4.6. *Lett $0 < \gamma < 2/L$. Under Assumption 4.1, $I - \gamma\nabla f$ is $\gamma L/2$-averaged.*

Proof. By the previous theorem, there exists a non-expansive operator R such that $L^{-1}\nabla f = (I + R)/2$. Thus, $I - \gamma\nabla f = (1 - \alpha)I + \alpha(-R)$ where $\alpha = \gamma L/2$, and the proof is completed by noticing that $-R$ is non-expansive. □

Theorem 4.7 (gradient algorithm)**.** *Let Assumption 4.1 hold true, and assume in addition that $\arg\min f \neq \emptyset$. Let $0 < \gamma < 2/L$. Then, each sequence (x^k) satisfying the recursion $x^{k+1} = x^k - \gamma\nabla f(x^k)$ converges to a minimizer of f.*

Proof. The assumption $\arg\min f \neq \emptyset$ ensures that $\mathrm{Fix}(I - \gamma\nabla f) \neq \emptyset$. The result is an immediate consequence of Theorem 4.3 and Lemma 4.6. □

4.3 The proximal point and the proximal gradient algorithms

4.3.1 The proximity operator

The proximity operator associated with a function f is the $\mathcal{X} \to \mathcal{X}$ mapping defined as

$$\mathrm{prox}_f(x) = \arg\min_{y\in\mathcal{X}} f(y) + \frac{1}{2}\|y - x\|^2 \tag{4.3.1}$$

provided this definition makes sense.

Proposition 4.8. *Let $f \in \Gamma_0(\mathcal{X})$. Then,*

(i) prox_f *is a well-defined mapping on $\mathcal{X}$.*

(ii) prox_f *is firmly non-expansive.*

(iii) $p = \mathrm{prox}_f(x) \Leftrightarrow x \in p + \partial f(p)$ *.*

Proof. Let us fix $x \in \mathcal{X}$. The convexity is preserved if we subtract $\frac{1}{2}\|\cdot\|^2$ from the function $f + \frac{1}{2}\|\cdot - x\|^2$. Thus, $f + \frac{1}{2}\|\cdot - x\|^2$ is a strongly convex $\Gamma_0(\mathcal{X})$ function. By Proposition 1.25, its $\arg\min$ is a singleton, which establishes the first point. Let p be the minimizer of this function. By Fermat's rule, $0 \in \partial(f + \frac{1}{2}\|\cdot - x\|^2)(p)$, thus, $0 \in \partial f(p) + p - x$ by Proposition 3.13. Consequently, $x - p \in \partial f(p)$, and the last point is established. Let $y \in \mathcal{X}$ and $q = \mathrm{prox}_f(y)$. Then, $y - q \in \partial f(q)$. This implies that $\langle (x - p) - (y - q), p - q\rangle \geq 0$ (see Exercise 1.2), or equivalently, that $\left\langle x - y, \mathrm{prox}_f(x) - \mathrm{prox}_f(y)\right\rangle \geq \|\mathrm{prox}_f(x) - \mathrm{prox}_f(y)\|^2$. By Proposition 4.2, the operator prox_f is firmly non-expansive. □

Exercise 4.1. Let C be a non empty, closed and convex set, and recall that the indicator function ι_C of the set C is the function in $\Gamma_0(\mathcal{X})$ defined as $\iota_C(x) = 0$ if $x \in C$ and $\iota_C(x) = \infty$ otherwise. Show that $\mathrm{prox}_{\gamma\iota_C(\cdot)}(x) = P_C(x)$ for each $\gamma > 0$, where we recall that P_C is the projector on C.

Corollary 4.9. *Let $f \in \Gamma_0(\mathcal{X})$. For every $\gamma > 0$, $\arg\min f = \mathrm{Fix}(\mathrm{prox}_{\gamma f})$.*

Proof. By Prop. 4.8, $x = \mathrm{prox}_{\gamma f}(x)$ if and only if $x \in x + \partial(\gamma f)(x)$, which reads $0 \in \partial(\gamma f)(x)$. By the Fermat rule, this is equivalent to say that x is a minimizer of γf, hence of f. □

The proof of the following proposition is an easy exercise.

Proposition 4.10. *Let $n \in \mathbb{N}_*$, and let $f_1, \ldots, f_n$ be functions in $\Gamma_0(\mathcal{X})$. For all $x = (x_1, \ldots, x_n) \in \mathcal{X}^N$, set $f(x) = f_1(x_1) + \cdots + f_n(x_n)$. Then, $f \in \Gamma_0(\mathcal{X}^N)$, and $\mathrm{prox}_f(x) = (\mathrm{prox}_{f_1}(x_1), \ldots, \mathrm{prox}_{f_n}(x_n))$.*

Proposition 4.11 (Moreau's identity)**.** *Consider $f \in \Gamma_0(\mathcal{X})$ and $\gamma > 0$. For every $x \in \mathbb{R}^n$,*

$$\operatorname{prox}_{\gamma f}(x) + \gamma \operatorname{prox}_{\gamma^{-1} f^*}(\frac{x}{\gamma}) = x\,. \tag{4.3.2}$$

Proof. Let $x \in \mathbb{R}^n$ and $p = \operatorname{prox}_{\gamma f}(x)$. Since p is defined as the solution of a convex optimization problem, Fermat's rule leads to $x - p \in \gamma \partial f(p)$. By Proposition 2.10, $p \in \partial f^*(\frac{x-p}{\gamma})$. Let us now denote $p^* = \frac{x-p}{\gamma}$. We have $\frac{x}{\gamma} - p^* = \frac{p}{\gamma} \in \gamma^{-1}\partial f^*(p^*)$. This is equivalent to $p^* = \operatorname{prox}_{\gamma^{-1} f^*}(\frac{x}{\gamma})$, so the conclusion follows by $\frac{x - \operatorname{prox}_{\gamma f}(x)}{\gamma} = \operatorname{prox}_{\gamma^{-1} f^*}(\frac{x}{\gamma})$. □

4.3.2 The proximal point algorithm

Our purpose is to find a minimizer of a function $g \in \Gamma_0(\mathcal{X})$. By Fermat's rule (Proposition 1.22), this amounts to finding a point $\bar{x}$ such that $0 \in \partial g(\bar{x})$. By Proposition 4.8–3., this inclusion reads $\bar{x} \in \operatorname{Fix} \operatorname{prox}_g$. The set of minimizers is not altered if we replace g with γg for any $\gamma > 0$. The **proximal point algorithm** reads:

$$x^{k+1} = \operatorname{prox}_{\gamma g}(x^k)\,. \tag{4.3.3}$$

The following theorem follows from Proposition 4.8 and Theorem 4.3.

Theorem 4.12 (proximal point algorithm)**.** *Let $g \in \Gamma_0(\mathcal{X})$ be such that $\arg\min g \neq \emptyset$. Then, given $\gamma > 0$, every sequence (x^k) verifying the recursion (4.3.3) converges towards a minimizer of g.*

Contrary to the gradient algorithm, the proximal point algorithm does not require that the function under study be smooth, and does not put any constraint on the step size $\gamma > 0$. On the other hand, the proximal point algorithm requires solving Problem (4.3.1) at each iteration, which can be computationally demanding.

4.3.3 The proximal gradient algorithm

In this paragraph, we consider two functions f and g such that f satisfies Assumption 4.1, and $g \in \Gamma_0(\mathcal{X})$. Our purpose is to find a minimizer of $f + g$. Observing that $\operatorname{dom} f = \mathcal{X}$ and using Proposition 3.13, this amounts to finding a point $\bar{x}$ such that $0 \in \nabla f(\bar{x}) + \partial g(\bar{x})$. Equivalently,

$$\bar{x} - \nabla f(\bar{x}) \in \bar{x} + \partial g(\bar{x})\,.$$

By Proposition 4.8, this inclusion can be read as $\bar{x} = \operatorname{prox}_g(\bar{x} - \nabla f(\bar{x}))$. We can extend this remark by observing that there is an identity between the minimizers of $f + g$ and those of $\gamma f + \gamma g$ for all $\gamma > 0$. In other words, we established the following property:

Proposition 4.13. *Let f, g two functions such that f satisfies Assumption 4.1, and such that $g \in \Gamma_0(\mathcal{X})$. Assume that $\arg\min(f + g) \neq \emptyset$. Then, $\bar{x} \in \arg\min(f + g)$ iff $\bar{x} = \operatorname{prox}_{\gamma g}(\bar{x} - \gamma \nabla f(\bar{x}))$.*

This proposition suggests the following algorithm, termed the **proximal gradient algorithm**:

$$x^{k+1} = \operatorname{prox}_{\gamma g}(x^k - \gamma \nabla f(x^k))\,. \tag{4.3.4}$$

Theorem 4.14 (Proximal gradient algorithm)**.** *Let f, g two functions such that f satisfies Assumption 4.1, and such that $g \in \Gamma_0(\mathcal{X})$. Assume that* $\arg\min(f+g) \neq \emptyset$. *Let $0 < \gamma < 2/L$. Then, every sequence (x^k) verifying the recursion (4.3.4) converges towards a minimizer of $f+g$.*

Proof. The operator $I - \gamma\nabla f$ is $\gamma L/2$-averaged by Lemma 4.6. The operator $\operatorname{prox}_{\gamma g}$ is firmly non-expansive, in other words $(1/2)$-averaged, by Proposition 4.8. Thus, the composition $\operatorname{prox}_{\gamma g}(I - \gamma\nabla f)$ is δ-averaged for some $\delta \in (0,1)$ by Lemma 4.4. Theorem 4.3 allows to conclude. □

The proximal gradient algorithm is often used when the computation of $\operatorname{prox}_{\gamma g}$ is easy. Examples are given here.

4.4 Applications

4.4.1 Projected gradient algorithm

Let $C \subset \mathcal{X}$ be a non empty closed convex set. We want to solve the problem

$$\inf_{x \in C} f(x)\,, \tag{4.4.1}$$

where the function f satisfies Assumption 4.1. Problem (4.4.1) is equivalent to the problem

$$\inf_{x \in \mathcal{X}} f(x) + \iota_C(x)\,.$$

We know from Exercise 4.1 that $\operatorname{prox}_{\gamma\iota_C} = P_C$ for each $\gamma > 0$. Consequently, the proximal gradient algorithm is

$$x^{k+1} = P_C(x^k - \gamma\nabla f(x^k))\,.$$

Under the assumptions of Theorem 4.14, this algorithm converges to a minimizer of $f + \iota_C$, *i.e.*, a minimizer of f on C.

4.4.2 Iterative soft-thresholding

Setting $\mathcal{X} = \mathbb{R}^n$ and letting $\eta > 0$, we want to solve the problem

$$\inf_{x \in \mathcal{X}} f(x) + \eta\|x\|_1 \tag{4.4.2}$$

where f satisfies Assumption 4.1, and where $\|x\|_1$ is the ℓ_1 of the vector x, defined as $\|x\|_1 = |x_1| + \cdots + |x_n|$ by writing $x = (x_1, \ldots, x_n)$.

Proposition 4.15. *The function* $\operatorname{prox}_{\eta|\,.\,|}$ *coincides with the so-called* soft thresholding *function, which is defined for all $x \in \mathbb{R}$ as:*

$$S_\eta(x) = \begin{cases} x - \eta & \text{if } x > \eta \\ 0 & \text{if } x \in [-\eta, \eta] \\ x + \eta & \text{if } x < -\eta\,. \end{cases}$$

Proof. Put $p = \operatorname{prox}_{\eta|\,.\,|}(x)$. By Proposition 4.8, $x \in p + \partial(\eta|\,.\,|_1)(p)$. In case $p > 0$, this implies after Example 1.1 that $x = p + \eta$, or equivalently $p = x - \eta > 0$. In case $p < 0$, we have $p = x + \eta < 0$. Finally, if $p = 0$, $x \in [-\eta, \eta]$. Thus, $p = S_\eta(x)$. □

From Proposition 4.10, we get that for all $x = (x_1, \cdots, x_n)$,

$$\mathrm{prox}_{\eta\|\,.\,\|_1}(x) = (S_\eta(x_1), \cdots, S_\eta(x_n))\,.$$

In this situation, the proximal gradient algorithm takes the form

$$\begin{aligned} y^k &= x^k - \gamma\nabla f(x^k) \\ x_i^{k+1} &= S_{\gamma\eta}(y_i^k) \qquad (\forall i = 1, \ldots, n), \end{aligned}$$

where $x^k = (x_1^k, \cdots, x_n^k)$ and $y^k = (y_1^k, \cdots, y_n^k)$. Under the assumptions of Theorem 4.14, the iterates x^k converge to a minimizer of (4.4.2).

4.5 Exercises

In the three following exercises, (x_n) is a sequence in $\mathcal{X}$, and S is a non-empty subset of $\mathcal{X}$. We denote as $\mathrm{acc}((x_n))$ the set of accumulation points of (x_n).

Exercise 4.2. Assume that the sequence $(\|x_n - x_\star\|)$ has a limit for each $x_\star \in S$, and furthermore, that $\mathrm{acc}((x_n)) \subset S$. Show that (x_n) converges to a point of S.

The following result is more general than the former:

Exercise 4.3. Assume that the sequence $(\|x_n - x_\star\|)$ has a limit for each $x_\star \in S$. Show that $\mathrm{acc}((x_n)) \cap S$ contains at most one point. In particular, if $\mathrm{acc}((x_n)) \subset S$, then (x_n) converges.

Exercise 4.4. Assume that the sequence (x_n) is bounded, and that S is non-empty, closed and convex. Assume that that there is $x_\star \in S$ such that $P_S(x_n) \to x_\star$ as $n \to \infty$. Show that $\mathrm{acc}((x_n)) \cap S \subset \{x_\star\}$. In particular, if $\mathrm{acc}((x_n)) \subset S$, then, $x_n \to x_\star$.

Exercise 4.5. We say that an operator $T : D \to \mathcal{X}$ is **quasi non-expansive** if

$$\forall x \in D,\ \forall y \in \mathrm{Fix}\, T, \quad \|Tx - y\| \le \|x - y\|.$$

This operator is said **strictly quasi non-expansive** if

$$\forall x \in D \setminus \mathrm{Fix}\, T,\ \forall y \in \mathrm{Fix}\, T, \quad \|Tx - y\| < \|x - y\|.$$

Let $T_1, T_2 : D \to D$ be two quasi non-expansive operators such that $\mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2 \neq \emptyset$. Assume that one of these operators at least is strictly quasi non-expansive. Show that

(i) $\mathrm{Fix}\, T_1 T_2 = \mathrm{Fix}\, T_1 \cap \mathrm{Fix}\, T_2$.

(ii) $T_1 T_2$ is quasi non-expansive.

Hint: To show (i), take $x \in \mathrm{Fix}\, T_1 T_2$ and distinguish the cases $x \in \mathrm{Fix}\, T_2$, $T_2 x \in \mathrm{Fix}\, T_1$ and the complementary case.

Exercise 4.6. Let S_1 and S_2 be two closed and convex sets such that $S_1 \cap S_2 \neq \emptyset$. Suggest an iterative algorithm involving the composition of two firmly non-expansive operators that converges to a point of $S_1 \cap S_2$, and prove this convergence. Use the previous exercise.

Chapter 5

Elements of monotone operator theory

Monotone operators can be seen as generalizations of the subdifferentials. They provide a theoretical framework for designing and studying many optimization algorithms. In the previous chapter, the convergence of some optimization algorithms was given a geometrical interpretation. This interpretation extends to the framework of monotone operators, providing powerful and elegant convergence proofs for a wider panel of algorithms.

5.1 Basic definitions and facts

5.1.1 Monotone and maximal monotone operators

In this chapter, an **operator** on the Euclidean space $\mathcal{X}$ is a mapping from $\mathcal{X}$ to $2^{\mathcal{X}}$ (sometimes we shall talk about a "set-valued operator"). The subdifferential of a function (see Definition 1.11) is a typical example of an operator. A single-valued operator, *i.e.*, a mapping A from $D \subset \mathcal{X}$ to $\mathcal{X}$ as in the previous chapter, is seen here as a particular case of a set-valued operator if we put $Ax = \emptyset$ when $x \notin D$. The image of x by an operator A will be arbitrarily denoted as $A(x)$ or Ax. The **domain** of A is the set $\text{dom}(A) = \{x \in \mathcal{X} : Ax \neq \emptyset\}$. The **image** of A is the set $\text{Im}(A) = \bigcup_{x \in \mathcal{X}} Ax$. The **graph** of A is the subset of $\mathcal{X} \times \mathcal{X}$ defined as $\text{gra}(A) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : y \in Ax\}$. It is obvious that an operator can be identified by its graph. As in Page 17, the inverse of A, denoted as A^{-1}, is the operator whose graph is $\text{gra}(A^{-1}) = \{(x, y) \in \mathcal{X} \times \mathcal{X} : (y, x) \in \text{gra}(A)\}$. Observe that $\text{dom}(A^{-1}) = \text{Im}(A)$. The **set of zeros** of an operator A is the set $\mathcal{Z}(A) = \{x \in \text{dom}(A) : 0 \in Ax\}$. The **composition** AB of the operator A with the operator B is the operator whose graph is $\text{gra}(AB) = \{(x, z) \in \mathcal{X} \times \mathcal{X} : \exists y \in \mathcal{X}, y \in Bx, z \in Ay\}$. Given a real number γ, we denote as γA the operator $(\gamma I)A$, where we recall that I is the identity operator. Finally, the **sum** $A + B$ of A and B is the operator $x \mapsto (A + B)(x) = \{y + z : y \in Ax, z \in Bx\}$ when $x \in \text{dom}(A) \cap \text{dom}(B)$, and $\emptyset$ elsewhere.

Definition 5.1. A set-valued operator A on the Euclidean space $\mathcal{X}$ is said **monotone** if for all $x, y \in \text{dom}(A)$ and for all $u \in Ax$ and $v \in Ay$, it holds that $\langle x - y, u - v \rangle \geq 0$.

In Exercise 1.2, we showed that the subdifferential of a function $f : \mathcal{X} \to [-\infty, \infty]$ is an example of a monotone operator.

Exercise 5.1. Show that if $\mathcal{X} = \mathbb{R}$ and if A is single-valued, then A is monotone *iff* A is non-decreasing.

The **resolvent** of an operator A is the operator $Q_A = (I + A)^{-1}$. Resolvents of monotone operators will be of prime importance in the remainder.

Proposition 5.1. *An operator A is monotone* iff *for all $\gamma > 0$, the operator $Q_{\gamma A}$ is a non-expansive operator from* $\text{Im}(I + \gamma A)$ *to* $\mathcal{X}$.

This proposition shows in particular that when A is monotone, $Q_{\gamma A}$ is single-valued. In other words, given $u \in \mathcal{X}$, the inclusion $u \in x + \gamma Ax$ admits at most one solution.

Proof. Observe that $x \in Q_{\gamma A}(u) \Leftrightarrow u = x + \gamma p$ with $p \in Ax$. Thus, the statement of the proposition can be rephrased as follows: A is monotone *iff* $\forall \gamma > 0$, $\forall x, y \in \text{dom}\, A$, $\forall p \in Ax$, $\forall q \in Ay$, $\|x - y\|^2 \leq \|x + \gamma p - y - \gamma q\|^2$.
We have

$$\|x + \gamma p - y - \gamma q\|^2 - \|x - y\|^2 = 2\gamma \langle x - y, p - q\rangle + \gamma^2 \|p - q\|^2.$$

If A is monotone, then the right hand side is obviously non negative. Conversely, if this term is non negative for all $\gamma > 0$, then dividing it by γ and making $\gamma \downarrow 0$, we get that $\langle x - y, p - q\rangle \geq 0$. □

Proposition 5.2. *The operator A is monotone* iff *for all $\gamma > 0$, the operator $Q_{\gamma A}$ is a firmly non-expansive operator from* $\text{Im}(I + \gamma A)$ *to* $\mathcal{X}$.

Proof. We only need to prove that if A is monotone, then $Q_{\gamma A}$ is firmly non expansive. Given $u, v \in \text{dom}\, Q_{\gamma A}$, let $x = Q_{\gamma A}(u)$ and $y = Q_{\gamma A}(v)$. Since $u - x \in \gamma Ax$ and $v - y \in \gamma Ay$, we get that $0 \leq \langle x - y, u - x - (v - y)\rangle = \langle x - y, u - v\rangle - \|x - y\|^2$ by the monotonicity of A. The result follows from Proposition 4.2. □

In this chapter, we shall study iterative algorithms for finding an element of $\mathcal{Z}(A)$ when A is a monotone operator. One motivation for doing this study is provided by the monotonicity of the subdifferential (Exercise 1.2) and by Fermat's rule (Proposition 1.22).
To this end, we first observe that $\mathcal{Z}(A) = \text{Fix}\, Q_{\gamma A}$ for any $\gamma > 0$. Second, the previous proposition shows that $Q_{\gamma A}$ is a firmly non expansive operator. Consequently, if the domain of $Q_{\gamma A}$ is the whole space $\mathcal{X}$, then the theorem of Krasnosel'skii Mann (Theorem 4.3) shows that for every initial value x_0, the sequence (x^k) produced by the algorithm $x^{k+1} = Q_{\gamma A}(x^k)$ converges to a point of $\mathcal{Z}(A)$. We thus need to guarantee that $\text{dom}\, Q_{\gamma A} = \mathcal{X}$.
Noting that the graph inclusion provides a partial ordering for the monotone operators, we make the following definition:

Definition 5.2. A monotone operator A on $\mathcal{X}$ is said **maximal** if its graph is a maximal element in the graph inclusion ordering.

Figure 5.1 is an illustration for a non-maximal and a maximal monotone operator on $\mathbb{R}$.

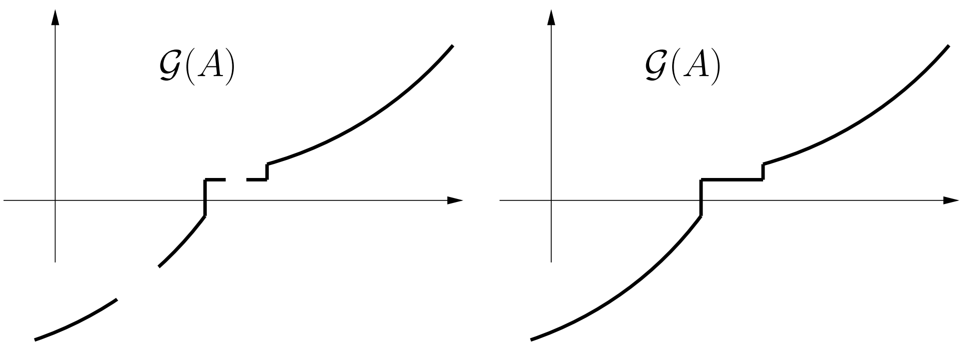

Figure 5.1: Non-maximal (left) *vs.* maximal (right) monotone operator on $\mathbb{R}$.

Exercise 5.2. Show that the maximal monotonicity of A, A^{-1}, and γA for $\gamma > 0$ are equivalent.

Exercise 5.3. Let A be a single-valued monotone operator with $\operatorname{dom} A = \mathcal{X}$. Show that if A is continuous, then it is maximal monotone.
Hint: Let $x, u \in \mathcal{X}$ be such that $\langle x - y, u - Ay \rangle \geq 0$ for all $y \in \mathcal{X}$. To show that $u = Ax$, work on the scalar product $\langle u - Ax, u - Az_\lambda \rangle$, where $z_\lambda = x + \lambda(u - Ax)$, and $\lambda > 0$.

Exercise 5.4. Show that a single-valued linear operator M is maximal monotone *iff* the symmetric operator $M + M^*$ is nonnegative (in the sense that $\langle x, (M + M^*)x \rangle \geq 0$ for all $x \in \mathcal{X}$).

Theorem 5.3 (Minty)**.** *A monotone operator A on $\mathcal{X}$ is maximal* iff $\operatorname{Im}(I + A) = \mathcal{X}$.

Equivalently, A is maximal *iff* $\operatorname{dom} Q_A = \mathcal{X}$.

Proof. Assume that $\operatorname{Im}(I + A) = \mathcal{X}$. Let B be a monotone operator such that $\operatorname{gra}(A) \subset \operatorname{gra}(B)$. Given $(x, u) \in \operatorname{gra}(B)$, it holds by assumption that there exists $y \in \operatorname{dom} A$ such that $x + u \in y + Ay$, thus, $x + u \in y + By$. On the other hand, $x + u \in x + Bx$. Consequently, $x = (I + B)^{-1}(x + u) = y$ by Proposition 5.1, and $u \in Ax$. Thus, $(x, u) \in \operatorname{gra}(A)$, and A is maximal.
The converse will be proven in Exercise 5.7. □

Using Propositions 5.1 and 5.2, we immediately get the following corollary of Theorem 5.3.

Corollary 5.4. *Given a set-valued operator A on $\mathcal{X}$, the following assertions are equivalent:*

(i) A is a maximal monotone operator.

(ii) For all $\gamma > 0$, $Q_{\gamma A}$ is a non expansive operator with domain $\mathcal{X}$.

(iii) For all $\gamma > 0$, $Q_{\gamma A}$ is a firmly non expansive operator with domain $\mathcal{X}$.

Corollary 5.5. *If $g \in \Gamma_0(\mathcal{X})$, then ∂g is a maximal monotone operator, and $Q_{\partial g} = \operatorname{prox}_g$.*

Proof. Combine Proposition 4.8 with the previous corollary. □

5.1.2 The proximal point algorithm

Along with Theorem 4.3, Corollary 5.4 leads to the following result.

Theorem 5.6 (proximal point algorithm)**.** *Let A be a maximal monotone operator on $\mathcal{X}$ such that $\mathcal{Z}(A) \neq \emptyset$. Then, for each $\gamma > 0$ and each initial value $x_0 \in \mathcal{X}$, the sequence (x^k) produced by the algorithm*

$$x^{k+1} = Q_{\gamma A}(x^k)$$

converges to a point of $\mathcal{Z}(A)$.

Corollary 5.5 shows that Theorem 4.12 is a particular case of Theorem 5.6.
We provide herein another instance of Theorem 5.6: the so-called *method of multipliers*.

.

Lemma 5.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two Euclidean spaces, let $f \in \Gamma_0(\mathcal{X})$, and let $M \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$.*

i) If f is coercive, then $M \triangleright f \in \Gamma_0(\mathcal{Y})$. Moreover, for each $y \in \mathcal{Y}$, the set

$$\arg\min_{u \in \mathcal{X}} f(u) + \frac{1}{2}\|Mu - y\|^2$$

is nonempty, and $\operatorname{prox}_{M \triangleright f}(y) = Mx$, where x is an arbitrary element of this set.

ii) If M is injective (which supposes $\dim(\mathcal{X}) \leq \dim(\mathcal{Y})$), then $M \triangleright f \in \Gamma_0(\mathcal{Y})$. Moreover, for each $y \in \mathcal{Y}$, the set

$$\arg\min_{u \in \mathcal{X}} f(u) + \frac{1}{2}\|Mu - y\|^2$$

is reduced to a singleton $\{x\}$, and $\operatorname{prox}_{M \triangleright f}(y) = Mx$.

Proof. We first establish (i). We first recall from Proposition 1.16 that $M \triangleright f$ is a convex function with the nonempty domain $\operatorname{dom}(M \triangleright f) = M \operatorname{dom}(f)$. Since f is coercive, $\inf f(\mathcal{X}) > -\infty$ by Proposition 1.23. Thus, $M \triangleright f$ is proper since $\inf(M \triangleright f)(\mathcal{Y}) \geq \inf f(\mathcal{X})$. Let us show that $M \triangleright f$ is l.s.c. We first show that if $y \in \operatorname{dom}(M \triangleright f)$, then $(M \triangleright f)(y)$ is attained by some x such that $y = Mx$. Indeed, $(M \triangleright f)(y) = \inf f + \iota_y(M\cdot)$. The function $f + \iota_y(M\cdot)$ is convex and proper, and by Proposition 1.21, it is l.s.c. Moreover, it is coercive, being larger than f. The result follows from Proposition 1.23. Consider now a sequence $y_k \to y \in \mathcal{Y}$, and let $c_k = (M \triangleright f)(y_k)$. To establish the lower semi continuity of $M \triangleright f$, we only need to consider the case where the sequence (c_k) is bounded. In this case, there exists x_k such that $Mx_k = y_k$ and $f(x_k) = c_k$. Consider an arbitrary sequence of such x_k. By the boundedness of (c_k) and the coercivity of f, the sequence (x_k) belongs to a compact (see, *e.g.*, the proof of Proposition 1.23), thus, it has accumulation points. Given an arbitrary subsequence $(\psi(k))$ such that $x_{\psi(k)} \to x^*$, we have $Mx^* = y$, and by the lower semi continuity of f, $\liminf f(x_{\psi(k)}) \geq f(x^*) \geq (M \triangleright f)(y)$. Consequently, $M \triangleright f \in \Gamma_0(\mathcal{Y})$.
The function

$$u \mapsto f(u) + \frac{1}{2}\|Mu - y\|^2 = f(u) + \frac{1}{2}\langle u, M^*Mu\rangle - \langle Mu, y\rangle + \frac{\|y\|^2}{2} \tag{5.1.1}$$

is clearly a coercive function in $\Gamma_0(\mathcal{X})$. Consequently, its $\arg\min$ is nonempty by Proposition 1.23. The last result can be checked by direct calculation.
We now establish (ii) by succinctly pointing out the differences with (i). Consider $y \in \operatorname{dom}(M \triangleright f)$. Since M is injective, $M \triangleright f(y) = f(M^{-1}y) > -\infty$, thus, $M \triangleright f$ is proper. To show that $M \triangleright f(y)$ is l.s.c., we similarly take a sequence $y_k \to y \in \mathcal{Y}$, and assume that the $c_k = (M \triangleright f)(y_k)$ are bounded. Here we simply have $c_k = f(x_k)$ where $x_k = M^{-1}y_k$. We now have $x_k \to M^{-1}y$, and the lower semicontinuity of $M \triangleright f$ follows from the lower semicontinuity of f. The function (5.1.1) is now the sum of a function in $\Gamma_0(\mathcal{X})$ and the strongly convex function $\langle u, M^*Mu\rangle/2$. Thus, it is a strongly convex function in $\Gamma_0(\mathcal{X})$. Consequently, its $\arg\min$ is reduced to a singleton by Proposition 1.25. The last result can also be checked by direct calculation. □

Given two Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, let $f \in \Gamma_0(\mathcal{X})$, $b \in \mathcal{Y}$, and $M \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$. We consider the minimization problem under affine equality constraints which is detailed in Section 3.2.2. In the framework of the Fenchel-Rockafellar duality theory, the primal problem is written

$$p = \inf_{\mathcal{X}} f + g \circ M \,,$$

where $g = \iota_{\{b\}} \in \Gamma_0(\mathcal{Y})$. Observing that $g^* = \langle b, \cdot \rangle$, the dual problem is

$$d = -\inf_{\mathcal{Y}} f^* \circ (-M^*) + \langle b, \cdot \rangle \ .$$

The set $\mathcal{S}$ of saddle points of the Lagrangian, whether it exists, is the set of points $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$ that satisfy the system of inclusions (3.2.8). If the qualification condition (3.2.9) holds true, then by Theorem 3.7, $p = d$, and the set of minimizers of the dual problem is not empty. We now observe that the primal problem can be written as $p = (M \triangleright f)(b)$. If f is coercive, then $M \triangleright f \in \Gamma_0(\mathcal{Y})$ by Lemma 5.7–(i). Obviously, the qualification condition (3.2.9) ensures that $b \in \text{dom}(M \triangleright f)$. Thus, the set of minimizers of the primal problem is not empty (*details to be provided*). This implies that $\mathcal{S} \neq \emptyset$ by Proposition 3.8–(iii). The method of multipliers is a technique for finding a point of $\mathcal{S}$. Let $\gamma > 0$. Starting with an arbitrary $\phi^0 \in \mathcal{Y}$, the algorithm consists in following iterations:

$$x^{k+1} \in \arg\min_{v \in \mathcal{X}} \left\{ f(v) + \left\langle Mv, \phi^k \right\rangle + \frac{\|Mv - b\|^2}{2\gamma} \right\} , \tag{5.1.2a}$$

$$\phi^{k+1} = \phi^k + \gamma^{-1} \left(Mx^{k+1} - b \right) . \tag{5.1.2b}$$

Theorem 5.8 (method of multipliers)**.** *Given two Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, let f be a coercive function in $\Gamma_0(\mathcal{X})$, let $b \in \mathcal{Y}$, and let $M \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$. Consider the minimization problem with affine equality constraints (3.2.7). Assume that the set $\mathcal{S}$ of saddle points for this problem,* i.e., *the points $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$ verifying (3.2.8), is not empty. Let $\gamma > 0$ be arbitrary. Consider the sequence of iterates $((x^k, \phi^k))$ produced by the algorithm (5.1.2). Then the sequence (ϕ^k) converges to $\phi^* \in \mathcal{Y}$, and the sequence (x^k) belongs to a compact set. Given any accumulation point x^* of (x^k), it holds that $(x^*, \phi^*) \in \mathcal{S}$.*

To prove this theorem, we show that the method of multipliers results from applying the proximal point algorithm to the dual problem $\inf_{\mathcal{Y}} f^* \circ (-M^*) + \langle b, \cdot \rangle$.

Proof. We show that the iterates

$$\phi^{k+1} = \text{prox}_{\gamma^{-1}(f^* \circ (-M^*) + \langle b, \cdot \rangle)}(\phi^k) \tag{5.1.3}$$

are those which are provided by Algorithm (5.1.2). Consider the function $F(y) = ((-M) \triangleright f)(y - b)$. We know from Exercise 2.5 that $((-M) \triangleright f)^* = f^* \circ (-M^*)$. Thus, $F^* = f^* \circ (-M^*) + \langle b, \cdot \rangle$. Moreover, $F \in \Gamma_0(\mathcal{Y})$ by Lemma 5.7–(i). Consequently, we can apply Moreau's identity (4.3.2) to obtain

$$\phi^{k+1} = \phi^k - \gamma^{-1} \text{prox}_{\gamma F}(\gamma \phi^k) \, . \tag{5.1.4}$$

Developing the prox operator at the right hand side, we get that

$$\begin{aligned} \text{prox}_{\gamma F}(\phi^k) &= \arg\min_{y \in \mathcal{Y}} ((-M) \triangleright f)(y - b) + \frac{1}{2\gamma} \|y - \gamma \phi^k\|^2 \\ &= b + \arg\min_{y \in \mathcal{Y}} ((-M) \triangleright f)(y) + \frac{1}{2\gamma} \|y + b - \gamma \phi^k\|^2 \\ &= b + \text{prox}_{\gamma((-M) \triangleright f)}(\gamma \phi^k - b) \, . \end{aligned}$$

Using Lemma 5.7–(i), we get that $\text{prox}_{\gamma((-M) \triangleright f)}(\gamma \phi^k - b) = -Mx^{k+1}$, where x^{k+1} satisfied the inclusion (5.1.2a), and ϕ^{k+1} is given by (5.1.2b).

Since $\mathcal{S} \neq \emptyset$, the dual problem $\inf f^* \circ (-M)^* + \langle b, \cdot \rangle$ has a minimizer, thus, the iterates ϕ^k which are given by Equation (5.1.3) converge to a dual solution ϕ^∞ by Theorem 4.12 (or Theorem 5.6). Equation (5.1.2a) can be rewritten as

$$x^{k+1} \in \arg\min_v \left\{ f(v) + \frac{\|Mv + \gamma\phi^k - b\|^2}{2\gamma} \right\} .$$

By the boundedness of (ϕ^k) and the coercivity of f, we obtain that the sequence (x^k) belongs to a compact. Moreover, considering Equation (5.1.4) and taking $k \to \infty$, we get by the continuity of the prox that $\operatorname{prox}_{\gamma F}(\gamma\phi^\infty) = 0$. This shows that $Mx^k \to b$, thus, any accumulation point x^∞ of (x^k) is a primal solution. □

5.2 Splitting algorithms

Let A and B be two maximal monotone operators such that $\mathcal{Z}(A+B) \neq \emptyset$. A splitting algorithm is an iterative algorithm for finding an element of $\mathcal{Z}(A+B)$ by performing operations that involve A and B separately. This requirement is usually unavoidable for obtaining implementable optimization algorithms. The remainder of this chapter is devoted to such algorithms.

5.2.1 The Douglas-Rachford splitting algorithm

The **Cayley transform** of a maximal monotone operator A is the mapping

$$C_A : \mathcal{X} \longrightarrow \mathcal{X}, \quad x \longmapsto 2Q_A - I .$$

The proof of the following proposition is left to the reader as an exercise:

Proposition 5.9. *C_A is a non expansive operator defined on $\mathcal{X}$.*

Given two maximal monotone operators A and B, the **Douglas-Rachford** (or Lions-Mercier) operator is the single-valued operator defined on $\mathcal{X}$ as

$$T_{A,B} = \frac{I + C_A C_B}{2} .$$

More explicitly,

$$T_{A,B}(x) = Q_A(2Q_B(x) - x) - Q_B(x) + x . \tag{5.2.1}$$

It is obvious from the definition of $T_{A,B}$ and from Proposition 5.9 that $T_{A,B}$ is firmly non-expansive.

Proposition 5.10. *Let A and B be two maximal monotone operators. Then, $\mathcal{Z}(A+B) \neq \emptyset \Leftrightarrow \operatorname{Fix} T_{A,B} \neq \emptyset$. If $\mathcal{Z}(A+B) \neq \emptyset$, then $Q_B(\operatorname{Fix} T_{A,B}) = \mathcal{Z}(A+B)$.*

Proof. We have the following equivalences:

$$\begin{aligned} x \in \operatorname{Fix} T_{A,B},\ y = Q_B(x) &\Leftrightarrow Q_A(2y - x) = y,\ y = Q_B(x) \\ &\Leftrightarrow 2y - x \in y + Ay,\ x \in y + By \\ &\Leftrightarrow 2y \in 2y + Ay + By,\ x \in y + By \\ &\Leftrightarrow y \in \mathcal{Z}(A+B),\ x \in y + By. \end{aligned}$$

□

This proposition, the firm non-expansiveness of $T_{A,B}$, and Theorem 4.3 lead to the following result:

Theorem 5.11 (Douglas-Rachford algorithm)**.** *Let A and B be two maximal monotone operators on $\mathcal{X}$ such that $\mathcal{Z}(A+B) \neq \emptyset$. Let $\gamma > 0$ and $x_0 \in \mathcal{X}$ be arbitrary, and consider the sequence (x^k) produced by the algorithm*

$$x^{k+1} = T_{\gamma A, \gamma B}(x^k)\,.$$

Then, $\operatorname{Fix} T_{\gamma A,\gamma B} \neq \emptyset$*, the sequence (x^k) converges to a point of* $\operatorname{Fix} T_{\gamma A,\gamma B}$*, and the sequence $(Q_{\gamma B}(x^k))$ converges to a point of $\mathcal{Z}(A+B)$.*

We shall describe two instances of this algorithm.

The Douglas-Rachford for minimizing the sum of two functions

Given two functions $f, g \in \Gamma_0(\mathcal{X})$, consider the problem $\inf_{\mathcal{X}} f + g$. Assume that $\mathcal{Z}(\partial f + \partial g) \neq \emptyset$. This is satisfied if the set of minimizers of the problem $\inf_{\mathcal{X}} f + g$ is non empty and if $0 \in \operatorname{ri}(\operatorname{dom} f - \operatorname{dom} g)$, as shown by Fermat's rule and by Proposition 3.13. The Douglas-Rachford algorithm considered here finds a point of $\mathcal{Z}(\partial f + \partial g)$. Given a step $\gamma > 0$, it consists in the following iterations, starting with an arbitrary $x^0 \in \mathcal{X}$.

$$\begin{aligned} z^{k+1} &= \operatorname{prox}_{\gamma g}(x^k)\,, \\ x^{k+1} &= \operatorname{prox}_{\gamma f}(2z^{k+1} - x^k) - z^{k+1} + x^k\,. \end{aligned} \tag{5.2.2}$$

The following theorem is a straightforward corollary of Theorem 5.11.

Theorem 5.12 (Douglas-Rachford minimization algorithm)**.** *Let $f, g \in \Gamma_0(\mathcal{X})$, and assume that $\mathcal{Z}(\partial f + \partial g) \neq \emptyset$. Let $\gamma > 0$ be arbitrary. Then each sequence (z^k) produced by Algorithm (5.2.2) converges to a point of $\mathcal{Z}(\partial f + \partial g)$.*

The Douglas-Rachford algorithm in the dual domain: ADMM

Given two Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\dim(\mathcal{X}) \leq \dim(\mathcal{Y})$, let $f \in \Gamma_0(\mathcal{X})$, $g \in \Gamma_0(\mathcal{Y})$, and let M be an injection in $\mathsf{L}(\mathcal{X}, \mathcal{Y})$. Consider the minimization problem

$$p = \inf_{\mathcal{X}} f + g \circ M\,.$$

Recalling the results of Section 3.2, the dual of this problem in the sense of the Fenchel-Rockafellar theory is

$$d = -\inf_{\mathcal{Y}} f^* \circ (-M^*) + g^*\,.$$

The set $\mathcal{S}$ of saddle points of the Lagrangian, whether it exists, is the set of points $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$ that satisfy the system of inclusions (3.2.5). We recall that if the qualification condition $0 \in \operatorname{ri}(M \operatorname{dom}(f) - \operatorname{dom}(g))$ holds true, then by Theorem 3.7, $p = d$, and the set of minimizers of the dual problem is not empty. If, furthermore, the set of minimizers of the primal problem $\inf f + g \circ M$ is not empty, then $\mathcal{S} \neq \emptyset$ by Proposition 3.8–(iii). The Alternating Direction Method of Mutlipliers (ADMM, see *e.g.*, Boyd et al. (2011)) is a popular algorithm for finding a saddle point for this problem. Let $\gamma > 0$ be some step size. Starting with $(z^0, \phi^0) \in \mathcal{Y}^2$, this algorithm is written:

$$x^{k+1} = \arg\min_{v\in\mathcal{X}} f(v) + \left\langle \phi^k, Mv \right\rangle + \frac{1}{2\gamma}\|Mv - z^k\|^2 \,, \tag{5.2.3a}$$

$$z^{k+1} = \arg\min_{w\in\mathcal{Y}} g(w) - \left\langle \phi^k, w \right\rangle + \frac{1}{2\gamma}\|Mx^{k+1} - w\|^2 \,, \tag{5.2.3b}$$

$$\phi^{k+1} = \phi^k + \gamma^{-1}\left(Mx^{k+1} - z^{k+1}\right) . \tag{5.2.3c}$$

In order to solve the problem $\min_x f(x) + g(Mx)$ that combines f and g in a complex manner, the ADMM requires only subproblems that involve f and g separately. For instance, we can solve the problem when f is quadratic and g has a simple proximal operator, even if the sum is not easy to deal with directly.

Theorem 5.13 (ADMM)**.** *Given two Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\dim\mathcal{X} \le \dim\mathcal{Y}$, let $f \in \Gamma_0(\mathcal{X})$, $g \in \Gamma_0(\mathcal{Y})$, and let M be an injective element of $\mathsf{L}(\mathcal{X},\mathcal{Y})$. Consider the minimization problem $\inf f + g \circ M$. Assume that the set $\mathcal{S}$ of saddle points for this problem,* i.e., *the points $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$ verifying the system of inclusions (3.2.5), is not empty. Let $\gamma > 0$ be arbitrary. Then, the sequence of iterates $((x^k, \phi^k))$ produced by the algorithm (5.2.3) converges to a point of $\mathcal{S}$.*

ADMM can be seen as an instance of the Douglas-Rachford algorithm applied to the dual problem $\inf f^* \circ (-M^*) + g^*$. We take this route to prove Theorem 5.13.

Proof. Define the maximal monotone operators $A = \partial(f^* \circ (-M^*))$ and $B = \partial g^*$. Noticing that $\mathrm{Im}(-M^*) = \mathcal{X}$ and applying Proposition 3.13 to the subdifferential $\partial(0+f^*\circ(-M^*))$, we get that $A = -M^*\partial f^* \circ (-M^*)$. Let $(x, \phi) \in \mathcal{S}$. By Proposition 2.10, Inclusion (3.2.5a) can be rewritten as $x \in \partial f^*(-M^*\phi)$. Plugging this into Inclusion (3.2.5b), we get that $0 \in A\phi + B\phi$, thus, $\mathcal{Z}(A+B) \neq \emptyset$. Now, let (u^k) be a sequence defined by the iterations $u^{k+1} = T_{\gamma^{-1}A,\gamma^{-1}B}(u^k)$. By Theorem 5.11, (u^k) converges to a point u^∞, and the sequence (ϕ^k), defined as $\phi^k = Q_{\gamma^{-1}B}(u^k)$, converges to $\phi^\infty \in \mathcal{Z}(A + B)$. Our first task is to show that (ϕ^k) is the one provided by Algorithm (5.2.3).
Since $\phi^k = Q_{\gamma^{-1}B}(u^k)$, we can write $u^k = \phi^k + \gamma^{-1}z^k$, where $z^k \in \partial g^*(\phi^k)$. Recalling Equation (5.2.1), let $y^{k+1} = Q_{\gamma^{-1}A}(2\phi^k - u^k) = \mathrm{prox}_{\gamma^{-1}f^*\circ(-M^*)}(\phi^k - \gamma^{-1}z^k)$. Then the output of the Douglas-Rachford algorithm is $u^{k+1} = y^{k+1} - \phi^k + u^k = y^{k+1} + \gamma^{-1}z^k$. Let us make explicit the expression of y^{k+1}. We know from Exercise 2.5 that $((-M) \rhd f)^* = f^* \circ (-M^*)$. Thus, using Moreau's identity (4.3.2), we get

$$y^{k+1} = \phi^k - \gamma^{-1}z^k - \gamma^{-1}\,\mathrm{prox}_{\gamma(-M)\rhd f}(\gamma\phi^k - z^k) \,.$$

Using Lemma 5.7–(ii), we can write

$$\mathrm{prox}_{\gamma(-M)\rhd f}(\gamma\phi^k - z^k) = -Mx^{k+1} \,, \tag{5.2.4}$$

where

$$x^{k+1} = \arg\min_{v\in\mathcal{X}} f(v) + \frac{1}{2\gamma}\|Mv + \gamma\phi^k - z^k\|^2 \,,$$

which is the right hand side of (5.2.3a). Replacing the value of y^{k+1} in the expression of u^{k+1}, we get $u^{k+1} = \phi^k - \gamma^{-1}z^k + \gamma^{-1}Mx^{k+1} + \gamma^{-1}z^k = \phi^k + \gamma^{-1}Mx^{k+1}$. We conclude by using that $\phi^{k+1} = Q_{\gamma^{-1}B}(u^{k+1}) = \mathrm{prox}_{\gamma^{-1}g^*}(\phi^k + \gamma^{-1}Mx^{k+1})$. Indeed, by Moreau's identity again,

$$\phi^{k+1} = \phi^k + \gamma^{-1}Mx^{k+1} - \gamma^{-1}z^{k+1} \,,$$

where

$$z^{k+1} = \mathrm{prox}_{\gamma g}(\gamma \phi^k + M x^{k+1}).$$

This equation and the previous one coincide with (5.2.3b) and (5.2.3c) respectively.
We now show that $((x^k, \phi^k))$ converges to a point of $\mathcal{S}$. Since $u^k \to u^\infty$ and $\phi^k \to \phi^\infty$, $z^k = \gamma(u^k - \phi^k)$ converges to a point z^∞. From Equation (5.2.3c), and the injectivity of M, the sequence (x^k) converges to x^∞, and $z^\infty = M x^\infty$. Using that $\phi^k = Q_{\gamma^{-1}\partial g^*}(\phi^k + \gamma^{-1} z^k)$ and taking k to ∞, we get that $\phi^\infty = Q_{\gamma^{-1}\partial g^*}(\phi^\infty + \gamma^{-1} z^\infty)$, which is equivalent to $z^\infty \in \partial g^*(\phi^\infty)$. Since $z^\infty = M x^\infty$, we get Inclusion (3.2.5b) with $(x, \phi) = (x^\infty, \phi^\infty)$. Using (5.2.4) and taking k to ∞, we obtain that $Q_{\gamma \partial((-M) \triangleright f)}(\gamma \phi^\infty + M x^\infty) = -M x^\infty$ which is equivalent to $\phi^\infty \in \partial((-M) \triangleright f)(-M x^\infty)$, or $-M x^\infty \in -M \partial f^*(-M^* \phi^\infty)$, which can be rewritten as Inclusion (3.2.5a). □

5.2.2 The Forward-Backward algorithm

Definition 5.3. Given $\alpha > 0$, a single-valued operator A defined on $\mathcal{X}$ is said α–**cocoercive** if αA is firmly non-expansive.

Exercise 5.5. Show that an α–cocoercive operator is maximal monotone.

Consider a convex and differentiable function f defined on $\mathcal{X}$. If ∇f is L-Lipschitz for some $L > 0$, then the Baillon-Haddad theorem (Theorem 4.5) shows that ∇f is L^{-1}–cocoercive.
Given a maximal monotone operator A, an α–cocoercive operator B and a real $\gamma > 0$, the **Forward-Backward** operator is the single-valued operator defined on $\mathcal{X}$ as

$$S_{\gamma A, \gamma B} = Q_{\gamma A}(I - \gamma B)\,.$$

The "forward" operator is $I - \gamma B$ while the "backward" operator is $Q_{\gamma A}$. Applying the latter consists in solving an implicit equation, hence the denomination.

Proposition 5.14. Fix $S_{\gamma A, \gamma B} = \mathcal{Z}(A + B)$. *Furthermore, if* $\gamma \in (0, 2\alpha)$, *then there exists* $\delta \in (0, 1)$ *such that* $S_{\gamma A, \gamma B}$ *is a* δ*–averaged operator.*

Proof. To obtain the first result, we write

$$x \in \mathcal{Z}(A + B) \Leftrightarrow 0 \in \gamma A x + \gamma B x \Leftrightarrow (I - \gamma B) x \in (I + \gamma A) x \Leftrightarrow x = (I + \gamma A)^{-1}(I - \gamma B) x\,.$$

Since B is α–cocoercive, there exists a non-expansive operator R such that $B = (2\alpha)^{-1} R + (2\alpha)^{-1} I$. Thus, the operator

$$I - \gamma B = \frac{\gamma}{2\alpha}(-R) + \Big(1 - \frac{\gamma}{2\alpha}\Big) I$$

is $\gamma/(2\alpha)$–averaged. Since $Q_{\gamma A}$ is $1/2$–averaged by Corollary–5.4, Lemma 4.4 leads to the second result. □

With the help of Theorem 4.3, this proposition immediately leads to:

Theorem 5.15 (Forward-Backward algorithm)**.** *Let A be a maximal monotone operator, and let B be a α-cocoercive operator on $\mathcal{X}$. Assume that $\mathcal{Z}(A + B) \neq \emptyset$. Let $\gamma \in (0, 2\alpha)$. Then, for each $x_0 \in \mathcal{X}$, the sequence (x^k) produced by the algorithm*

$$x^{k+1} = S_{\gamma A, \gamma B}(x^k)$$

converges to a point of $\mathcal{Z}(A + B)$.

Observe that this theorem generalizes Theorem 4.14: the forward-backward algorithm is a general version of the proximal gradient algorithm.
In the next paragraph, we describe another instance of the forward-backward algorithm: the so-called Vũ-Condat algorithm.

The Vũ-Condat primal-dual algorithm

Let $\mathcal{X}$ and $\mathcal{Y}$ be two Euclidean spaces. Let f be a convex and differentiable function defined on $\mathcal{X}$, and assume that ∇f is L–Lipschitz. Let $g \in \Gamma_0(\mathcal{X})$ and $h \in \Gamma_0(\mathcal{Y})$. Given $M \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$, consider the minimization problem

$$\inf_{\mathcal{X}} f + g + h \circ M \,,$$

The dual problem for this problem is

$$\inf_{\mathcal{Y}} (f+g)^* \circ (-M^*) + h^* \,.$$

Recalling the inclusions (3.2.5), the set $\mathcal{S}$ of saddle points for this problem, whether it exists, satisfies the following conditions:

$$0 \in \nabla f(x) + \partial g(x) + M^* \phi \tag{5.2.5a}$$
$$0 \in -Mx + \partial h^*(\phi) \tag{5.2.5b}$$

(notice indeed that $\partial(f+g) = \nabla f + \partial g$, since f and g satisfy the assumptions of Proposition 3.13). By Theorem 3.7 and Proposition 3.8, the condition

$$0 \in \mathrm{ri}(M \,\mathrm{dom}(g) - \mathrm{dom}(h)) \,,$$

along with a non emptyness condition of the set of minimizers of the primal problem, ensures that $\mathcal{S} \neq \emptyset$.
Given two step sizes $\tau, \gamma > 0$, the Vũ-Condat algorithm is an algorithm for finding a point of $\mathcal{S}$. It consists in the following iterations:

$$z^{k+1} = \operatorname*{arg\,min}_{v \in \mathcal{Y}} h(v) - \left\langle v, \phi^k \right\rangle + \frac{1}{2\gamma} \|v - Mx^k\|^2 \,, \tag{5.2.6a}$$
$$\phi^{k+1} = \phi^k + \gamma^{-1}(Mx^k - z^{k+1}) \,, \tag{5.2.6b}$$
$$x^{k+1} = \operatorname*{arg\,min}_{w \in \mathcal{X}} g(w) + \left\langle w, \nabla f(x^k) \right\rangle + \left\langle w, M^*(2\phi^{k+1} - \phi^k) \right\rangle + \frac{1}{2\tau} \|w - x^k\|^2 \,. \tag{5.2.6c}$$

The Vũ-Condat algorithm allows an even finer splitting than the ADMM. It uses more structure in the problem in order to reduce the per-iteration complexity. We can deal separately with differentiable functions, proximable functions and linear operators, while guaranteeing convergence of the iterates.

Theorem 5.16 (Vũ-Condat algorithm)**.** *Let f be a convex and differentiable function defined on $\mathcal{X}$. Assume that ∇f is L–Lipschitz for some $L \geq 0$. Let $g \in \Gamma_0(\mathcal{X})$, $h \in \Gamma_0(\mathcal{Y})$, and $M \in \mathsf{L}(\mathcal{X}, \mathcal{Y})$. Consider the minimization problem $\inf_{\mathcal{X}} f + g + h \circ M$, and assume that the set $\mathcal{S}$ of saddle points of this problem,* i.e., *the set of points $(x, \phi) \in \mathcal{X} \times \mathcal{Y}$ verifying the inclusions (5.2.5), is not empty. Consider the iterations (5.2.6), and assume that the positive numbers τ and γ are such that $\tau^{-1} - \gamma^{-1}\|M\|^2 > L/2$, where $\|\cdot\|$ is the spectral norm. Then the sequence of the iterates (x^k, ϕ^k) produced by the algorithm (5.2.6) converges to a point of $\mathcal{S}$.*

The remainder of this section is devoted to the proof of this theorem. We first observe that if $L = 0$ (*i.e.*, f is affine), we can always replace L with $\varepsilon > 0$ small enough so that $\tau^{-1} -$

$\gamma^{-1}\|M\|^2 > \varepsilon/2$. Thus, we can always assume that $L > 0$, as we shall do hereinafter.

We first introduce some new notations. We denote as $\mathcal{W}$ the Euclidean space $\mathcal{X} \times \mathcal{Y}$ endowed with the scalar product $\langle (x,\phi), (y,\psi) \rangle = \langle x, y \rangle + \langle \phi, \psi \rangle$, where $x, y \in \mathcal{X}$, $\phi, \psi \in \mathcal{Y}$, and the scalar products $\langle x, y \rangle$ and $\langle \phi, \psi \rangle$ are those of $\mathcal{X}$ and $\mathcal{Y}$ respectively (even though we use the same notations for scalar products and norms in these different spaces, their use will be always clear from the context). In the remainder, when we denote an element $u \in \mathcal{W}$ as $u = (x, \phi)$, we mean that $x \in \mathcal{X}$ and $\phi \in \mathcal{Y}$. The following matrix notation for some operators on $\mathcal{W}$ will also be convenient: given the set-valued operators $C_{xx} : \mathcal{X} \to 2^{\mathcal{X}}$, $C_{xy} : \mathcal{X} \to 2^{\mathcal{Y}}$, $C_{yx} : \mathcal{Y} \to 2^{\mathcal{X}}$, and $C_{yy} : \mathcal{Y} \to 2^{\mathcal{Y}}$, the operator

$$C = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix}$$

is the $\mathcal{W} \to 2^{\mathcal{W}}$ operator defined as $C((x,\phi)) = \{(y,\psi) \in \mathcal{W} : y \in C_{xx}x + C_{xy}\phi, \psi \in C_{yx}\, x + C_{yy}\psi\}$. .

Now, define on $\mathcal{W}$ the operators

$$A = \begin{bmatrix} \partial g & M^* \\ -M & \partial h^* \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \nabla f & 0 \\ 0 & 0 \end{bmatrix},$$

where 0 denotes generically an operator whose image is the set $\{0\}$. With these notations, the inclusion (5.2.5) is rewritten as $0 \in Au + Bu$. Let us inspect the basic properties of A and B.

Exercise 5.6. Given two maximal monotone operators C and D defined respectively on $\mathcal{X}$ and $\mathcal{Y}$, show that $\begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$ is maximal monotone on $\mathcal{W}$.

This exercise shows that the operator $\begin{bmatrix} \partial g & 0 \\ 0 & \partial h^* \end{bmatrix}$ is maximal monotone. We also know by Exercise 5.4 that the linear operator $\begin{bmatrix} & M^* \\ -M & \end{bmatrix}$ is maximal monotone. In these conditions, it can be shown that the operator A, which is the sum of these two operators, is maximal monotone (see *e.g.*, (Brézis, 1973, Lemma 2.4). Another proof that is specific to our context will be given as an exercise).
Turning to the operator B, we notice that

$$B = \nabla F, \quad \text{where} \quad F : \mathcal{W} \to \mathbb{R},\ (x, \phi) \mapsto f(x). \tag{5.2.7}$$

By Baillon-Haddad's theorem, B is a L^{-1}–cocoercive operator.

We just showed that A and B satisfy the assumptions of Theorem 5.15. Therefore, one first idea for solving our minimization problem is to implement the Forward-Backward algorithm $u^{k+1} = Q_{\rho A}(I - \rho B)u^k$ for some $\rho > 0$. However, due to the structure of A, implementing the operator $Q_{\rho A}$ is difficult because the operations involving g and h^* would have to be performed jointly (check this). What is needed is an algorithm that performs operations on g and h *separately.*

The idea goes as follows. Given $\tau > 0$ and $\gamma > 0$, define the linear operator

$$V = \begin{bmatrix} \tau^{-1}I & M^* \\ M & \gamma I \end{bmatrix}.$$

The inclusion $0 \in Au + Bu$ can be rewritten as $(V - B)u \in (V + A)u$. Observe now that the operator

$$V + A = \begin{bmatrix} \partial g + \tau^{-1} I & 2M^* \\ 0 & \partial h^* + \gamma I \end{bmatrix}$$

has an "upper triangular" structure. Thanks to this structure, we shall obtain an implementable and convergent algorithm if some care is taken in the choice of V.
The inclusion $(V - B)u \in (V + A)u$ may suggest the iterative algorithm

$$(V - B)u^k \in (V + A)u^{k+1} . \tag{5.2.8}$$

This inclusion can be written equivalently $(I - V^{-1}B)u^k \in (I + V^{-1}A)u^{k+1}$, provided V is invertible. Below, we shall show that the iteration

$$u^{k+1} = (I + V^{-1}A)^{-1}(I - V^{-1}B)u^k \tag{5.2.9}$$

is well defined for all u^k, and is equivalent to (5.2.8). Moreover, provided $\tau^{-1} - \gamma^{-1}\|M\|^2 > L/2$, the resulting algorithm is an instance of the Forward-Backward algorithm whose convergence is established by Theorem 5.15. The proof will be done in three steps:

Step 1: We show that when $\gamma\tau^{-1} > \|M\|^2$, the linear symmetric operator V is positive definite. This defines a new scalar product $\langle u, w\rangle_V = \langle u, Vw\rangle$ in the space $\mathcal{W}$. We denote as $\mathcal{W}_V$ the space $\mathcal{W}$ endowed with this new scalar product, and we denote as $\|\cdot\|_V$ the associated norm.

Step 2: In the space $\mathcal{W}_V$, the operator $V^{-1}A$ is maximal monotone, $V^{-1}B$ is cocoercive, and the operator defined by Equation (5.2.9) satisfies the assumptions of Theorem 5.15. Consequently, the sequence (u^k) is well defined, and it converges to a point of $\mathcal{Z}(A + B)$.

Step 3: Getting back to the inclusion (5.2.8), we make use of the upper triangular structure of $V + A$ to obtain Algorithm (5.2.6). We show that the sequence of iterates $((x^k, \phi^k))$ converges to a point of $\mathcal{S}$.

The first step is achieved by the following lemma:

Lemma 5.17. *The symmetric operator V is positive definite if $\gamma\tau^{-1} > \|M\|^2$.*

This condition is obviously satisfied when $\tau^{-1} - \gamma^{-1}\|M\|^2 > L/2$ as in the statement of Theorem 5.16.

Proof. Given an arbitrary non zero vector $u = (x, \phi) \in \mathcal{W}$, we have

$$\begin{aligned} \langle u, Vu\rangle &= \tau^{-1}\|x\|^2 + \gamma\|\phi\|^2 + 2\langle \phi, Mx\rangle \\ &\geq \tau^{-1}\|x\|^2 + \gamma\|\phi\|^2 - 2\|M\|\,\|\phi\|\,\|x\| \\ &= (\tau^{-1} - \gamma^{-1}\|M\|^2)\|x\|^2 + (\gamma^{1/2}\|\phi\| - \gamma^{-1/2}\|M\|\,\|x\|)^2 . \end{aligned} \tag{5.2.10}$$

Using the inequality $\tau^{-1} - \gamma^{-1}\|M\|^2 > 0$, and dealing separately with the cases $x \neq 0$ and $x = 0$, we get that $\langle u, Vu\rangle > 0$, hence the result. □

We now turn to the second step.

Lemma 5.18. *The operator $V^{-1}A$ is maximal monotone in $\mathcal{W}_V$.*

Proof. Given any $u, v \in \mathrm{dom}(A)$ and any $p \in Au$, $q \in Av$, we have $\langle u - v, V^{-1}p - V^{-1}q\rangle_V = \langle u - v, p - q\rangle \geq 0$, hence the monotonicity of $V^{-1}A$ in $\mathcal{W}_V$. The maximality of $V^{-1}A$ is deduced from the maximality of A, as it can be checked by the reader as an exercise. □

Lemma 5.19. *Given any vector $v = (x, 0) \in \mathcal{W}$, we have*

$$\langle v, V^{-1}v\rangle \leq (\tau^{-1} - \gamma^{-1}\|M\|^2)^{-1}\|x\|^2.$$

Moreover, for any $w = (x, \phi) \in \mathcal{W}$, we have

$$\left(\tau^{-1} - \gamma^{-1}\|M\|^2\right)\|x\|^2 \leq \langle w, Vw\rangle \ .$$

Proof. It is well known that any invertible matrix C with a block decomposition $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{bmatrix}$ where C_{11} is a square matrix satisfies $[C^{-1}]_{11} = (C_{11} - C_{12}C_{22}^{-1}C_{21})^{-1}$, where $[C^{-1}]_{11}$ is the upper left block of C^{-1} having the same size as C_{11}. Applying this identity to the linear operator V, we get that $\langle v, V^{-1}v\rangle = \langle x, (\tau^{-1} - \gamma^{-1}M^*M)^{-1}x\rangle$. Since $M^*M \leq \|M\|^2 I$ in the positive semidefinite ordering of the symmetric matrices, we have $(\tau^{-1} - \gamma^{-1}M^*M)^{-1} \leq (\tau^{-1} - \gamma^{-1}\|M\|^2)^{-1}I$, hence the first result.
The second result is deduced immediately from the inequality (5.2.10). □

Lemma 5.20. *The operator $V^{-1}B$ is κ/L–cocoercive in $\mathcal{W}_V$, where $\kappa = \tau^{-1} - \gamma^{-1}\|M\|^2$.*

Proof. Given $u = (x, \phi), v = (y, \psi) \in \mathcal{W}$, we have

$$\begin{aligned}
\|V^{-1}Bu - V^{-1}Bv\|_V^2 &= \langle Bu - Bv, V^{-1}(Bu - Bv)\rangle \\
&= \langle (\nabla f(x) - \nabla f(y), 0), V^{-1}(\nabla f(x) - \nabla f(y), 0)\rangle \\
&\leq \kappa^{-1}L^2\|x - y\|^2 \\
&\leq \kappa^{-2}L^2\|u - v\|_V^2
\end{aligned}$$

by the L–Lipschitz property of ∇f and by Lemma 5.19. We now recall from (5.2.7) that $B = \nabla F$ in $\mathcal{W}$. Thus, $V^{-1}B = \nabla F$ in $\mathcal{W}_V$, and the result follows from Baillon-Haddad's theorem. □

When $\kappa > L/2$, all the assumptions of Theorem 5.15 are satisfied, and each sequence of iterates (u^k) produced by the algorithm (5.2.9) converges to a point of $\mathcal{Z}(V^{-1}A + V^{-1}B) = \mathcal{Z}(A + B)$ in $\mathcal{W}_V$. Since the topologies of $\mathcal{W}$ and $\mathcal{W}_V$ are equivalent, this convergence takes place also in $\mathcal{W}$, which concludes Step 2.
We now turn to step 3, making the inclusion (5.2.8) explicit. Recall the expression of $V + A$ above, and observe that

$$V - B = \begin{bmatrix} \tau^{-1}I - \nabla f & M^* \\ M & \gamma I\end{bmatrix}.$$

Writing $u^k = (x^k, \phi^k)$, the inclusion (5.2.8) is written

$$\begin{bmatrix} I - \tau\nabla f & \tau M^* \\ \gamma^{-1}M & I\end{bmatrix}\begin{bmatrix} x^k \\ \phi^k\end{bmatrix} \in \begin{bmatrix} \tau\partial g + I & 2\tau M^* \\ 0 & \gamma^{-1}\partial h^* + I\end{bmatrix}\begin{bmatrix} x^{k+1} \\ \phi^{k+1}\end{bmatrix},$$

or equivalently,

$$\begin{aligned}
\phi^{k+1} &= \mathrm{prox}_{\gamma^{-1}h^*}(\gamma^{-1}Mx^k + \phi^k), \\
x^{k+1} &= \mathrm{prox}_{\tau g}(x^k - \tau\nabla f(x^k) + \tau M^*(\phi^k - 2\phi^{k+1})).
\end{aligned} \tag{5.2.11}$$

The second equation coincides with (5.2.6c). By Moreau's identity (4.3.2), the first equation can be rewritten as

$$\phi^{k+1} = \phi^k + \gamma^{-1} M x^k - \gamma^{-1} \operatorname{prox}_{\gamma h}(M x^k + \gamma \phi^k)\,,$$

which amounts to (5.2.6a)–(5.2.6b).
We now show that the limit (x^∞, ϕ^∞) of the sequence $((x^k, \phi^k))$ is a saddle point. Using Equations (5.2.11) and taking k to infinity, we obtain $\phi^\infty = Q_{\gamma^{-1}\partial h^*}(\gamma^{-1} M x^\infty + \phi^\infty)$ and $x^\infty = Q_{\tau\partial g}(x^\infty - \tau \nabla f(x^\infty) - \tau M^* \phi^\infty)$. One can see that these equations are equivalent to the inclusions (5.2.5). Theorem 5.16 is proven.

Remark 5.1. The Vũ-Condat algorithm described here can be straightforwardly generalized to the case where ∇f is replaced with a general cocoercive operator, and where ∂g and ∂h^* are replaced with general maximal monotone operators.

5.2.3 Discussion

We conclude by commenting briefly the algorithms we introduced in this chapter.
Considering the Vũ-Condat algorithm, we first make the following observations:

- If $h \circ M = 0$, we recover the proximal gradient algorithm. Indeed, in this case, we can assume without generality loss that $M = 0$ and $h = 0$. The primal and dual problems become decoupled, and so is the case of the inclusions (5.2.5). The iterations (5.2.6c) coincide with the proximal gradient iterations (4.3.4), and Theorem 4.14 is encompassed by Theorem 5.16.

- In the case where $f = 0$ and $M = I$, we recover the Douglas-Rachford algorithm if we set $\tau = \gamma$. Indeed, in this case, Equations (5.2.11) become

$$\phi^{k+1} = \operatorname{prox}_{\gamma^{-1}h^*}(\gamma^{-1} x^k + \phi^k)\,, \tag{5.2.12}$$
$$x^{k+1} = \operatorname{prox}_{\gamma g}(x^k + \gamma(\phi^k - 2\phi^{k+1}))\,. \tag{5.2.13}$$

  Write $s^k = x^k + \gamma \phi^k$, and let

$$y^{k+1} = \operatorname{prox}_{\gamma h}(s^k)\,. \tag{5.2.14}$$

  Then, applying Moreau's identity to the right hand side of Equation (5.2.12), we get that $\gamma\phi^{k+1} = s^k - y^{k+1}$. Plugging this equation into (5.2.13), we get

$$\begin{aligned} s^{k+1} &= x^{k+1} + \gamma\phi^{k+1} = \operatorname{prox}_{\gamma g}(x^k + \gamma\phi^k - 2(s^k - y^{k+1})) + s^k - y^{k+1} \\ &= \operatorname{prox}_{\gamma g}(2y^{k+1} - s^k) + s^k - y^{k+1}\,, \end{aligned} \tag{5.2.15}$$

  and it remains to compare Equations (5.2.14)–(5.2.15) with Equations (5.2.2) to obtain the result.

  Notice that when $f = 0$ and $M = I$, the assumption $\tau = \gamma$ is not covered by the statement of Theorem 5.16. However, the proof of this theorem can be adapted to include this case. Details are provided in Condat (2013).

We now provide some observations on the computational complexity of the splitting algorithms studied above. An important observation in this respect is that the Douglas-Rachford algorithm and ADMM require the implementation of proximity operators related with both functions involved in the minimization problem. On the other hand, when one of these functions (namely f) is smooth, the proximal gradient and the Vũ-Condat algorithms contend themselves with the gradient of this function. The computation of the proximity operator is sometimes demanding.

As an example, assume $f(x) = 0.5\|Ax - b\|^2$, where A is a matrix with large dimensions, as it is frequently the case in the fields of statistical learning and large scale optimization. Then, the computation of $\nabla f(x) = A^*(Ax - b)$ only requires a matrix-vector multiplication. On the other hand, since $\text{prox}_{\gamma f}(x)$ is the unique solution of the equation $\gamma \nabla f(v) + v = x$, it is written as

$$\text{prox}_{\gamma f}(x) = (\gamma A^* A + I)^{-1}(x + \gamma A^* b).$$

Before starting the algorithm, the inversion of the matrix $\gamma A^* A + I$ is required. The algorithm can be implemented only if this matrix inversion is affordable. Fast inversion algorithms exist only when A has a certain structure (Toeplitz, circulant, isometric, etc.). For general matrices, algorithms based on the computation of ∇f are often preferred.
Another point concerns the impact of the operator M on the computational complexity. In the context of ADMM, the iteration (5.2.3a) requires solving an inclusion of the type

$$\gamma \partial f(v) + M^* M v \in \cdots$$

If the operator $M^* M$ has no particular structure, the solution of this inclusion can be computationally demanding. This problem is avoided by the Vũ-Condat algorithm, where the computational impact of M on the iterations (5.2.6) is limited to matrix-vector multiplications.

5.3 Exercises

Exercise 5.7 (Fitzpatrick function and proof of Minty's theorem)**.** This exercise is devoted to the long part of the proof of Minty's theorem: if A is a maximal monotone operator on a Euclidean space $\mathcal{X}$, then $\text{Im}(I + A) = \mathcal{X}$.

Let A be a set-valued operator on a Euclidean space $\mathcal{X}$, and let $\mathcal{G}(A)$ be its graph, assumed to be nonempty. The *Brézis-Haraux* function associated with A is

$$\begin{array}{rccl} \Phi_A & : & \mathcal{X} \times \mathcal{X} & \longrightarrow & (-\infty, \infty] \\ & & (x, u) & \longmapsto & \sup_{(y,v) \in \mathcal{G}(\mathcal{A})} \langle x - y, v - u \rangle \,, \end{array}$$

and the *Fitzpatrick* function associated with this operator is

$$\begin{array}{rccl} \Psi_A & : & \mathcal{X} \times \mathcal{X} & \longrightarrow & (-\infty, \infty] \\ & & (x, u) & \longmapsto & \Phi_A(x, u) + \langle x, u \rangle \ , \end{array}$$

in other words,

$$\Psi_A(x, u) = \sup_{(y,v) \in \mathcal{G}(\mathcal{A})} \langle x, v \rangle - \langle y, v \rangle + \langle y, u \rangle \ .$$

We start by showing that these functions characterize the maximal monotonicity of the operator A.

1. Show that A is monotone *iff* $\Phi_A = 0$ on $\mathcal{G}(\mathcal{A})$.
2. Show that $\Phi_A(x, u) \leq 0$ if and only if $\{(x, u)\} \cup \mathcal{G}(\mathcal{A})$ is the graph of a monotone operator.
3. Deduce from the last question that a monotone operator A is maximal *iff* $\Phi_A(x, u) > 0$ whenever $(x, u) \notin \mathcal{G}(\mathcal{A})$.

We shall assume in the remainder that A is a maximal monotone operator. The maximality of A can be characterized by the statement just shown. Equivalently,

$$A \text{ is maximal} \quad \textit{iff} \quad \Psi_A(x,u) > \langle x,u\rangle \text{ whenever } (x,u) \notin \mathcal{G}(\mathcal{A})\,. \tag{5.3.1}$$

It will be more convenient to make use of this last result because Ψ_A is convex, which Φ_A is not in general.

4. Show that Ψ_A is a proper Fenchel-Legendre transform whose expression can be provided.
5. Let f be a convex and proper function on $\mathcal{X}$ that satisfies $f + \|\cdot\|^2/2 \geq 0$. Show that there exists $\phi \in \mathcal{X}$ such that
$$\forall x \in \mathcal{X}, \quad \langle x, \phi\rangle - f(x) + \frac{\|\phi\|^2}{2} \leq 0\,,$$
and deduce that
$$\forall x \in \mathcal{X}, \quad f(x) + \frac{\|x\|^2}{2} \geq \frac{\|\phi + x\|^2}{2}\,.$$
Hint: Use duality.
6. Show that $\Psi_A + \|\cdot\|^2/2 \geq 0$, and deduce that
$$\exists (y,v) \in \mathcal{X}\times\mathcal{X},\ \forall (x,u) \in \mathcal{X}\times\mathcal{X},\ \Psi_A(x,u) + \frac{\|(x,u)\|^2}{2} \geq \frac{\|(x,u)+(v,y)\|^2}{2}\,.$$

In the remainder, (y,v) will be a point that satisfies the previous statement.

7. Show that
$$\begin{aligned}\forall (x,u) \in \mathcal{G}(A),\ \langle x,u\rangle &\geq \frac{\|v\|^2}{2} + \langle x,v\rangle + \frac{\|y\|^2}{2} + \langle y,u\rangle \\ &\geq -\langle y,v\rangle + \langle x,v\rangle + \langle y,u\rangle\ ,\end{aligned}$$
and deduce that $v \in Ay$.
8. Replacing (x,u) with (y,v) in the first inequality above, show that $v = -y$, thus, that $0 \in \mathrm{Im}(I+A)$.
9. Given an arbitrary $z \in \mathcal{X}$, show that $z \in \mathrm{Im}(I+A)$.
Hint: Apply the previous result to a properly defined maximal monotone operator.

Chapter 6

The non convex case: Introduction to the Clarke subdifferential

In this chapter, we make a foray in the world of non-smooth and non-convex functions, introducing a new type of subdifferential called the Clarke subdifferential, which reduces to the convex subdifferential introduced in Chapter 1 when the function under study is convex.
Our starting point is to reinterpret the convex subdifferential in terms of a duality result involving the so-called directional derivative. We then generalize this picture to the non-convex case. The remainder of our exposition is heavily inspired from (Clarke et al., 1998, Chap. 2).

6.1 The directional derivative

Definition 6.1. Let $f : \mathcal{X} \to (-\infty, \infty]$, let $x \in \operatorname{dom} f$, assumed non empty, and let $d \in \mathcal{X}$. The **directional derivative** of f at x in the direction d is

$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

provided this limit exists in $[-\infty, \infty]$.

One prominent particular case is the case where $f'(x; d)$ exists for each $d \in \mathcal{X}$ and is linear in d, *i.e.*, the case where there exists a vector denoted as $\nabla f(x)$ and called the (Gâteaux) gradient of f at x such that $f'(x; d) = \langle \nabla f(x), d \rangle$. We then say that f is (Gâteaux) differentiable. We shall get back to this case later.
We focus here on the convex case.

Proposition 6.1 (directional derivatives in the convex case)**.** *Let $f : \mathcal{X} \to (-\infty, \infty]$ be proper and convex, and let $x \in \operatorname{dom} f$. Then,*

1. *For each $d \in \mathcal{X}$, the function*

$$\begin{array}{cccc} \varphi & : & (0, \infty) & \longrightarrow & (-\infty, \infty] \\ & & t & \longmapsto & \dfrac{f(x + td) - f(x)}{t} \end{array}$$

   is non decreasing, thus, $f'(x; d)$ exists.

2. *If $x \in \operatorname{dom} \partial f$, then, $f'(x; \cdot)$ is proper, and*

$$\phi \in \partial f(x) \quad \Leftrightarrow \quad \langle \phi, \cdot \rangle \leq f'(x; \cdot).$$

Proof. Let $d \in \mathcal{X}$, and put $0 < t_1 < t_2$. If $f(x+t_2 d) = \infty$, then the result is obvious. Otherwise, writing $\lambda = t_1/t_2$, it holds that $f(x+t_1 d) = f(\lambda(x+t_2 d)+(1-\lambda)x) \leq \lambda f(x+t_2 d)+(1-\lambda)f(x)$ by the convexity of f. By rearranging these terms, we obtain that $\varphi(t_1) \leq \varphi(t_2)$.
To establish (2), observe that

$$\phi \in \partial f(x) \;\Leftrightarrow\; \forall t > 0, \forall d \in \mathcal{X}, \langle \phi, d \rangle \leq \frac{f(x+td) - f(x)}{t}$$

by the definition of the subdifferential. Taking $t \downarrow 0$, we get that $\phi \in \partial f(x) \Rightarrow \langle \phi, \cdot \rangle \leq f'(x; \cdot)$. Conversely, we know from Item (1) that

$$f'(x; y-x) = \inf_{t>0} \frac{f(x+t(y-x)) - f(x)}{t} \leq f(y) - f(x)$$

by taking $t = 1$. Thus, if $\langle \phi, y-x \rangle \leq f'(x; y-x)$ for an arbitrary $y \in \mathcal{X}$, then, $\langle \phi, y-x \rangle \leq f(y) - f(x)$, which shows that $\phi \in \partial f(x)$. □

Thus, in the convex case, the subdifferential can be completely determined by the directional derivative. In fact, Proposition 6.1–(2) can be interpreted in terms of a duality between the directional derivative and the subdifferential of a convex function (this point will be detailed and made more precise below in a more general context). We shall generalize this idea to the functions which are not necessarily convex. In order to define our new subdifferential, we shall need a more ubiquitous object than the directional derivative. The notion of support function will play a central role in our exposition.

6.2 The support function

Definition 6.2. The **support function** of a nonempty set $S \subset \mathcal{X}$ is the function

$$\begin{array}{rcl} \mathcal{X} & \longrightarrow & (-\infty, \infty] \\ \phi & \longmapsto & \sup\{\langle \phi, x \rangle \,:\, x \in S\} \end{array}$$

It is obvious from this definition that the support function coincides with ι_S^*, the Fenchel-Legendre transform of the indicator function ι_S of the set S.

Exercise 6.1. Establish the following facts:

1. If S is non empty, then $\iota_S^* \in \Gamma_0(\mathcal{X})$.

2. If $S \subset \mathcal{X}$ is non empty and closed, then, S is compact *iff* ι_S^* is bounded on the unit ball of $\mathcal{X}$. Moreover, $\|S\| \leq L$, where L is the bound of ι_S^* on the unit ball.

3. If $S \subset \mathcal{X}$ is non empty, convex and closed, then, S is completely determined by ι_S^* by the fact that
$$x \in S \quad \Leftrightarrow \quad \forall \phi \in \mathcal{X}, \langle \phi, x \rangle \leq \iota_S^*(\phi).$$
Suggest two variants of the proof, one based on the separation theorem (Proposition 1.4), and the other based on the Fenchel-Moreau theorem (Theorem 2.5).

4. If $S, S' \subset \mathcal{X}$ are non empty, closed and convex, then,
$$S \subset S' \quad \Leftrightarrow \quad \iota_S^* \leq \iota_{S'}^*.$$

One useful property of the support functions is that the Minkowski sum of two sets is converted into the sum of their support functions. The proof of the following proposition is left as an exercise.

Proposition 6.2. *Let $S, S' \subset \mathcal{X}$ be non empty, and recall that $S + S'$ is the set $\{x + x' : x \in S, x' \in S'\}$. Then, $\iota^*_{S+S'} = \iota^*_S + \iota^*_{S'}$.*

Given a non empty set $S \subset \mathcal{X}$, it is obvious from the definition of the support function (see also the proof below) that $\iota^*_S(\lambda\phi) = \lambda\iota^*_S(\phi)$ for each $\lambda > 0$, in other words, ι^*_S is **positively homogeneous**. It turns out that this property is enough for a proper Fenchel-Legendre transform to be the support function of a closed convex set:

Theorem 6.3. *A function $g \in \Gamma_0(\mathcal{X})$ is the support function of a non empty, closed, and convex set* iff *g is positively homogeneous.*

Proof. We first observe that the indicators are the only functions such that $f = \lambda f$ for each $\lambda > 0$. Moreover, $g = g^{**}$ by the Fenchel-Moreau theorem. Thus, we need to show that $g^* = \lambda g^*$ for each $\lambda > 0$ *iff* g is positively homogeneous. We have

$$\begin{aligned}\lambda g^*(x) &= \lambda \sup_\phi \langle \phi, x\rangle - g(\phi) = \lambda \sup_\phi \left\langle \lambda^{-1}\phi, x\right\rangle - g(\lambda^{-1}\phi) \\ &= \sup_\phi \langle \phi, x\rangle - \lambda g(\lambda^{-1}\phi) = (\lambda g(\lambda^{-1}\cdot))^*(x),\end{aligned}$$

hence the result. ☐

6.3 The generalized directional derivative and the Clarke subdifferential

In all the remainder of this chapter, we shall work on a function $f : \mathcal{D} \to \mathbb{R}$ where $\mathcal{D}$ is an open domain of $\mathcal{X}$, and assume without further mention that f is **locally Lipschitz** on $\mathcal{D}$, *i.e.*, for each $x \in \mathcal{D}$, there exist two constants $\varepsilon > 0$ and $L > 0$ that may depend on x, and that satisfy

$$\forall y, z \in B_{\mathcal{X}}(x, \varepsilon), \ \|f(y) - f(z)\| \le L\|y - z\|$$

(we recall that $B_{\mathcal{X}}(x, \varepsilon)$ is the open ball with center x and radius ε, and we assume that ε is small enough so that $B_{\mathcal{X}}(x, \varepsilon) \subset \mathcal{D}$).

6.3.1 Definitions and basic properties

Definition 6.3. The **generalized derivative** of f at x in the direction $d \in \mathcal{X}$ is defined as

$$f^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}.$$

Proposition 6.4. *The following properties hold true:*

1. *$\|f^\circ(x; d)\| \le L\|d\|$, where L is the Lipschitz constant alluded to above.*
2. *The function $d \mapsto f^\circ(x; d)$ is positively homogeneous and convex.*
3. *The function $(x, d) \mapsto f^\circ(x; d)$ is upper semi-continuous.*

Proof. For each $y \in \mathcal{D}$ close enough to x, $d \in \mathcal{X}$, and $t > 0$ small enough, $|f(y+td) - f(y)| \leq tL\|d\|$, hence the first result.
Set $\lambda > 0$. Then,

$$f^\circ(x; \lambda d) = \lambda \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + t\lambda d) - f(y)}{t\lambda} = \lambda f^\circ(x; d).$$

Set $\lambda \in (0, 1)$, and let $d_1, d_2 \in \mathcal{X}$. Then,

$$\begin{aligned} f^\circ(x; \lambda d_1 + (1-\lambda)d_2) &= \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + t(\lambda d_1 + (1-\lambda)d_2)) - f(y + t\lambda d_1) + f(y + t\lambda d_1) - f(y)}{t} \\ &\leq \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + t(1-\lambda)d_2) - f(y)}{t} + \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + t\lambda d_1) - f(y)}{t} \\ &= f^\circ(x; \lambda d_1) + f^\circ(x; (1-\lambda)d_2) = \lambda f^\circ(x; d_1) + (1-\lambda) f^\circ(x; d_2), \end{aligned}$$

and the second result is shown.
We now need to show that $\limsup_{(x_n, d_n) \to (x,d)} f^\circ(x_n; d_n) \leq f^\circ(x; d)$. Since

$$f^\circ(x_n; d_n) = \limsup_{\substack{y \to x_n \\ t \downarrow 0}} \frac{f(y + td_n) - f(y)}{t},$$

we can always find y_n close to x_n and $t_n > 0$ close to zero such that

$$\begin{aligned} \|y_n - x_n\| + t_n &< \frac{1}{n}, \quad \text{and} \\ f^\circ(x_n; d_n) - \frac{1}{n} &< \frac{f(y_n + t_n d_n) - f(y_n)}{t_n}. \end{aligned}$$

Since f is locally Lipschitz near x, we have

$$\begin{aligned} \frac{f(y_n + t_n d_n) - f(y_n)}{t_n} &= \frac{f(y_n + t_n d) - f(y_n)}{t_n} + \frac{f(y_n + t_n d_n) - f(y_n + t_n d)}{t_n} \\ &\leq \frac{f(y_n + t_n d) - f(y_n)}{t_n} + L\|d_n - d\|. \end{aligned}$$

Making $n \to \infty$, we get from these inequalities that $\limsup f^\circ(x_n; d_n) \leq f^\circ(x; d)$. $\square$

In conjunction with Theorem 6.3, this proposition shows that $f^\circ(x; \cdot)$ is the support function of a non empty, closed and convex set, hence the following definition.

Definition 6.4 (Clarke subdifferential)**.** For each $x \in \mathcal{D}$, the set which support function is $f^\circ(x; \cdot)$ is the **Clarke subdifferential** of f at x. This set is denoted $\partial f(x)$.

Even though we use the same notation for the Clarke subdifferential and the subdifferential introduced by Definition 1.11, the nature of the subdifferential at hand will always be clear from the context.
Definition 6.4 calls for the following remarks:

1. The Clarke dubdifferential is a **local** object, contrary to the subdifferential introduced by Definition 1.11.

2. $\partial f(x)$ is non empty, convex and compact, and furthermore, $\|\partial f(x)\| \leq L$.

3. $\iota^*_{\partial f(x)} = f^\circ(x;\cdot)$, and

$$\phi \in \partial f(x) \quad \Leftrightarrow \forall d \in \mathcal{X},\ \langle \phi, d\rangle \leq f^\circ(x;d).$$

4. If $(x_n, \phi_n) \in \mathcal{D} \times \mathcal{X}$ are such that $\phi_n \in \partial f(x_n)$ and $(x_n, \phi_n) \to_n (x, \phi) \in \mathcal{D} \times \mathcal{X}$, then $\phi \in \partial f(x)$.

The reader may check the items 1 to 3 from the results we established on the support functions and the generalized derivatives. The last item can be proven by noticing that for each $d \in \mathcal{X}$, we have

$$\langle \phi, d\rangle = \lim_n \langle \phi_n, d\rangle \leq \limsup_n f^\circ(x_n; d) \leq f^\circ(x; d),$$

where the second inequality is due to Proposition 6.4–3.

Exercise 6.2. Let $\mathcal{D} = \mathbb{R}$ and $f(x) = x \vee 0$. Prove that $f^\circ(0; d) = d \vee 0$ and that $\partial f(0) = [0, 1]$.

Exercise 6.3. Given an integer $m > 0$, let $\mathcal{D} = \mathbb{R}^m$ and $f(x) = \|x\|$. Prove that $f^\circ(0; d) = \|d\|$ and that $\partial f(0) = \mathrm{cl}(B_{\mathbb{R}^m}(0, 1))$.

Exercise 6.4. Let $\mathcal{D} = \mathbb{R}$ and $f(x) = -|x|$. Compute $f^\circ(0; d)$ and $\partial f(0)$. Compute the directional derivative $f'(0; d)$. Why is this function not the support function of a convex set ?

Exercise 6.5. If the function f has a local maximum or a local minimum at $x \in \mathcal{D}$, then, $0 \in \partial f(x)$.

Exercise 6.6. Show that $f^\circ(x; -d) = (-f)^\circ(x; d)$.

6.3.2 Basic calculus rules

Proposition 6.5. *For all $\lambda \in \mathbb{R}$, it holds that $\partial(\lambda f) = \lambda \partial f$.*

Proof. If $\lambda \geq 0$, then $(\lambda f)^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \dfrac{\lambda f(y + td) - \lambda f(y)}{t} = \lambda f^\circ(x; d)$, thus, $\partial(\lambda f) = \lambda \partial f$. Assume now that $\lambda = -1$. Then, $(-f)^\circ(x; d) = f^\circ(x; -d)$. Consequently,

$$\begin{aligned}
\zeta \in \partial(-f)(x) &\Leftrightarrow \forall d, \langle \zeta, d\rangle \leq (-f)^\circ(x; d) \\
&\Leftrightarrow \forall d, \langle -\zeta, -d\rangle \leq f^\circ(x; -d) \\
&\Leftrightarrow \forall d, \langle -\zeta, d\rangle \leq f^\circ(x; d) \\
&\Leftrightarrow -\zeta \in \partial f(x) \\
&\Leftrightarrow \zeta \in -\partial f(x).
\end{aligned}$$

More generally, if $\lambda < 0$, then

$$\zeta \in \partial(-|\lambda| f)(x) \Leftrightarrow \zeta \in -\partial(|\lambda| f)(x) \Leftrightarrow \zeta \in -|\lambda| \partial f(x),$$

and the proposition is proven. □

Proposition 6.6. *Let $g : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz. Then, $\partial(f + g)(x) \subset \partial f(x) + \partial g(x)$.*

Proof. From the definition of f°, it is easy to see that $(f + g)^\circ(x; \cdot) \leq f^\circ(x; \cdot) + g^\circ(x; \cdot)$. The result follows from Proposition 6.2 and Exercise 6.1–4. □

Compare this proposition with Proposition 3.13.

Exercise 6.7. Given an integer $m > 0$, let $g_1, \ldots, g_m$ be locally Lipschitz functions on a common open set $\mathcal{D} \subset \mathcal{X}$, and let $\lambda_1, \ldots, \lambda_m$ be scalars. Show that

$$\partial \left(\sum_{i=1}^{m} \lambda_i g_i \right) \subset \sum_{i=1}^{m} \lambda_i \partial g_i.$$

6.3.3 Relation with the gradient

Getting back to Definition 6.1, we recall that when the directional derivative $f'(x; d)$ exists for each $d \in \mathcal{X}$ and is linear in d, *i.e.*, when there exists a vector $\nabla f(x) \in \mathcal{X}$ such that

$$\forall d \in \mathcal{X}, \ \frac{f(x+td) - f(x)}{t} \xrightarrow[t \downarrow 0]{} \langle \nabla f(x), d \rangle \,, \tag{6.3.1}$$

then f is said **differentiable** at x, and the vector $\nabla f(x)$ is called the **gradient** of f at x. In this section, we study some of the relations between this gradient and $\partial f(x)$.

Remark 6.1. In most generality, when the convergence (6.3.1) holds true for a given general function g in place of f, this function is said *Gâteaux differentiable* at x. The function g is said differentiable (or more specifically, *Fréchet differentiable*) when this convergence is uniform while d belongs to the unit ball of $\mathcal{X}$. In our situation, this distinction is irrelevant; Since our function f is locally Lipschitz at x, these two types of convergence can be proven to be equivalent, and you are encouraged to do this proof as an exercise.

Proposition 6.7. *The following hold true.*

1. *If f is differentiable at x, then, $\nabla f(x) \in \partial f(x)$.*
2. *If f is $\mathcal{C}^1$ at x,* i.e., *if f is differentiable with a continuous gradient at x, then, $\{\nabla f(x)\} = \partial f(x)$.*

Proof. It is clear that the directional derivative $f'(x; d)$, whether it exists, satisfies $f'(x; d) \leq f^\circ(x; d)$. If f is differentiable at x, then, $f'(x; d) = \langle \nabla f(x), d \rangle \leq f^\circ(x; d)$ for each $d \in \mathcal{X}$, hence the first result.
Assume that $f \in \mathcal{C}^1$ at x, and let y be near x and $t > 0$ be small. By the mean value theorem, there exists z in the line segment $(y, y + td)$ such that

$$\frac{f(y+td) - f(y)}{t} = \langle \nabla f(z), d \rangle \,.$$

Taking $t \to 0$ and using the continuity of ∇f at x, we obtain that $f^\circ(x; d) = \langle \nabla f(x), d \rangle$. Thus, letting $\zeta \in \partial f(x)$, we get that $\langle \zeta, d \rangle \leq \langle \nabla f(x), d \rangle$ for all $d \in \mathcal{X}$. This is true if and only if $\zeta = \nabla f(x)$. □

You can compare this proposition with Proposition 1.19.

6.3.4 Convex and regular functions

We saw in Proposition 1.10 above that a convex function is continuous in the interior of its domain. What's more, we have the following result that we admit in these notes (the interested reader may refer to (Rockafellar, 2015, Th. 10.4) for a proof):

Proposition 6.8. *A convex function g is locally Lipschitz at $x \in \operatorname{dom} g$ iff $x \in \operatorname{int}(\operatorname{dom} g)$.*

Remembering Proposition 6.1–2, the following proposition asserts that the convex and the Clarke subdifferentials coincide at any point in the interior of the domain of a convex function.

Proposition 6.9. *Assume that g is convex, and that $x \in \operatorname{int}(\operatorname{dom} g)$. Then, $g'(x; \cdot) = g^\circ(x; \cdot)$.*

Proof. Let $L > 0$ be the Lipschitz norm of g in a neighborhood of x, which existence is guaranteed by Proposition 6.8. For $\delta > 0$ small enough, we can write

$$g^\circ(x; d) = \lim_{\varepsilon \downarrow 0} \sup_{y : \|y-x\| \le \delta\varepsilon} \sup_{t : 0 < t < \varepsilon} \frac{g(y+td) - g(y)}{t} = \lim_{\varepsilon \downarrow 0} \sup_{y : \|y-x\| \le \delta\varepsilon} \frac{g(y+\varepsilon d) - g(y)}{\varepsilon},$$

where the second equality is justified by Proposition 6.1–1 and the convexity of g. Moreover,

$$\left| \frac{g(y+\varepsilon d) - g(y)}{\varepsilon} - \frac{g(x+\varepsilon d) - g(x)}{\varepsilon} \right| = \left| \frac{g(y+\varepsilon d) - g(x+\varepsilon d)}{\varepsilon} - \frac{g(y) - g(x)}{\varepsilon} \right| \le 2L\delta.$$

By consequence, $g^\circ(x; d) \le g'(x; d) + 2L\delta$. Since δ is arbitrary, we obtain that $g^\circ(x; d) \le g'(x; d)$. The inequality $g'(x; d) \le g^\circ(x; d)$ is obvious, hence the result. □

We just showed that the convex functions are regular in the sense of the following definition.

Definition 6.5 (regular functions)**.** The locally Lipschitz function f is said **regular** at x if $f'(x; d)$ exists for all $d \in \mathcal{X}$, and if $f'(x; d) = f^\circ(x; d)$.

By Proposition 6.7–2, the $\mathcal{C}^1$ functions provide another class of regular functions.

Exercise 6.8. Assume $\mathcal{D} = \mathbb{R}$. Show that $f(x) = |x|$ is regular at $x = 0$ while $f(x) = -|x|$ is not.

Under the regularity, the inclusion in Proposition 6.6 becomes an identity:

Proposition 6.10 (subdifferential of the sum of regular functions)**.** *Assume that the locally Lipschitz functions f and g are regular at x. Then, $\partial(f+g)(x) = \partial f(x) + \partial g(x)$. Moreover, $f + g$ is regular at x.*

Proof. By Proposition 6.6, we only need to prove that $\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$. For each $d \in \mathcal{X}$, we have

$$\begin{aligned}
\max\{\langle \zeta + \theta, d \rangle \,:\, \zeta \in \partial f(x), \theta \in \partial g(x)\} &= f^\circ(x; d) + g^\circ(x; d) \\
&= f'(x; d) + g'(x; d) \\
&= (f+g)'(x; d) \quad \text{(see Definition 6.1)} \\
&\le (f+g)^\circ(x; d) \\
&= \max\{\langle \zeta, d \rangle \,:\, \zeta \in \partial(f+g)(x)\},
\end{aligned}$$

thus, these two maxima are equal, and, furthermore, $(f+g)^\circ(x; \cdot) = (f+g)'(x; \cdot)$. □

6.3.5 The gradient formula

It is well-known in analysis that any locally Lipschitz function is differentiable almost everywhere (see, *e.g.*, (Rockafellar and Wets, 1998, Chap. 9), (Clarke et al., 1998, §2.8 and Chap. 3)). The following theorem is useful in that it provides a simple way of obtaining the Clarke subdifferential of a function at x from the gradients of this function near x. To better see this, you may redo Exercises 6.2 and 6.3 and the first part of Exercise 6.4 by applying this theorem.
We denote as $\mathrm{conv}(S)$ the convex hull of the set $S \subset \mathcal{X}$, which is the smallest convex set for inclusion that contains S.

Theorem 6.11. *Let $\Omega \subset \mathcal{X}$ be any set of Lebesgue measure zero, and let Ω_f be the set of points where f is not differentiable. Then,*

$$\partial f(x) = \mathrm{conv}\{\lim \nabla f(x_i) \,:\, x_i \to x,\, x_i \notin \Omega \cup \Omega_f\}.$$

In this statement, $\lim \nabla f(x_i)$ represents an accumulation point, *i.e.*, the limit of a converging subsequence $(\nabla f(x_i))$ as $x_i \to x$ and f is differentiable at any of the x_i's. One consequence of the theorem is that it is not necessary to consider *all* such subsequences. Any set of Lebesgue measure zero can be avoided without affecting the result.

Proof. We know from the remark 2 that follows Definition 6.4 that $\|\partial f(y)\| \le L < \infty$ for y sufficiently close to x. Moreover, $\nabla f(x_i) \in \partial f(x_i)$ by Proposition 6.7–1. Thus, the set $\{\nabla f(x_i) \,:\, x_i \to x,\, x_i \notin \Omega \cup \Omega_f\}$ is bounded. Consequently, the closed set $\{\lim \nabla f(x_i) \,:\, x_i \to x,\, x_i \notin \Omega \cup \Omega_f\}$ is compact. By Bolzano-Weierstrass, all the accumulation points $\lim \nabla f(x_i)$ belong to this set. Moreover, if $\phi_i \in \partial f(x_i)$ and $(x_i, \phi_i) \to (x, \phi)$, it follows from the remark 4 that follows Definition 6.4 that $\phi \in \partial f(x)$. Thus, $\{\lim \nabla f(x_i) \,:\, x_i \to x,\, x_i \notin \Omega \cup \Omega_f\} \subset \partial f(x)$, and even more, writing $A = \mathrm{conv}\{\lim \nabla f(x_i) \,:\, x_i \to x,\, x_i \notin \Omega \cup \Omega_f\}$ we have $A \subset \partial f(x)$ by the convexity of $\partial f(x)$.
It remains to prove the reverse inclusion. Equivalently, we prove that $f^\circ(x;\cdot) = \iota^*_{\partial f(x)} \le \iota^*_A$ by applying the result of Exercise 6.1–4. Setting $d \in \mathcal{X}$, we shall show that for each $\varepsilon > 0$, it holds that $f^\circ(x;d) - \varepsilon \le \iota^*_A(d)$. Note that

$$\iota^*_A(d) = \sup_{\zeta \in A} \langle \zeta, d\rangle = \limsup_{y \to x, y \notin \Omega \cup \Omega_f} \langle \nabla f(y), d\rangle\,.$$

Let $\delta > 0$ be small enough so that f is Lipschitz on $B(x,\delta)$, and $\langle \nabla f(z), d\rangle \le \iota^*_A(d) + \varepsilon$ for each $z \in B(x,\delta)$, $z \notin \Omega \cup \Omega_f$. Given $y \in B(x, \delta/2)$, define the line segment

$$L_y = \left\{ y + td, 0 < t < \frac{\delta}{2\|d\|} \right\}.$$

We show that for almost all $y \in B(x,\delta/2)$, the Lebesgue measure of $L_y \cap (\Omega \cup \Omega_f)$ on the line segment L_y is zero. Indeed, setting $\mathbf{1}_C(x) = 1$ if x belongs to the set C and 0 otherwise, this Lebesgue measure is

$$\int_{(0,\delta/(2\|d\|)} \mathbf{1}_{\Omega\cup\Omega_f}(y + sd)\; ds. \tag{6.3.2}$$

Denoting as $\lambda(\cdot)$ the Lebesgue measure on the ball $B(x,\delta)$, we have by Fubini's theorem that

$$\begin{aligned}\int_{B(x,\delta/2)} dy \int_{(0,\delta/(2\|d\|)} ds\; \mathbf{1}_{\Omega\cup\Omega_f}(y + sd) &= \int_{(0,\delta/(2\|d\|)} ds \int_{B(x,\delta/2)} dy\; \mathbf{1}_{\Omega\cup\Omega_f}(y + sd)\\ &\le \int_{(0,\delta/(2\|d\|)} ds\; \lambda(B(x,\delta) \cap (\Omega \cup \Omega_f))\\ &= 0,\end{aligned}$$

thus, the integral (6.3.2) is zero for almost all $y \in B(x, \delta/2)$. Choose one such y in $B(x, \delta/2)$. Since the function $t \mapsto f(y + td) - f(y)$ is absolutely continuous on $[0, \delta/(2\|d\|)]$ with an a.e. derivative $\langle \nabla f(y + td), d\rangle$, we have

$$f(y + td) - f(y) = \int_0^t \langle \nabla f(y + sd), d\rangle \; ds, \quad 0 < t < \frac{\delta}{2\|d\|}.$$

But since $y + sd \in B(x, \delta)$, we have $\langle \nabla f(y + sd), d\rangle \leq \iota_A^*(d) + \varepsilon$, thus,

$$\frac{f(y + td) - f(y)}{t} \leq \iota_A^*(d) + \varepsilon,$$

except possibly when y belongs to a set of Lebesgue measure zero. In fact, this inequality is true for each $y \in B(x, \delta/2)$ thanks to the continuity of f. Consequently, $f^\circ(x; d) - \varepsilon \leq \iota_A^*(d)$, which concludes the proof. □

The following corollary is deduced straightforwardly.

Corollary 6.12. $f^\circ(x; d) = \limsup_{y \to x, y \notin \Omega \cup \Omega_f} \langle \nabla f(y), d\rangle.$

Notations

$B_{\mathcal{X}}(a,r)$	:	Open ball with center a and radius r in the metric space $\mathcal{X}$
$\mathrm{cl}(C)$	:	Closure of the set C
$\mathrm{conv}(C)$	:	Convex hull of the set S
$\dim(\mathcal{V})$	:	Dimension of the vector space $\mathcal{V}$
$\mathrm{dom}(A)$	:	Domain of the set-valued operator A
$\mathrm{dom}(f)$	:	Domain of the function f
$\mathrm{epi}(f)$	:	Epigraph of the function f
f^*	:	Fenchel-Legendre transform of the function f
$f \,\square\, g$	:	Infimal convolution of the functions f and g
$\mathrm{Fix}(T)$	:	Set of fixed points of the operator T
$\mathrm{int}(C)$	:	Interior of the set C
l.s.c.	:	Lower semicontinous
$M \triangleright f$	:	Infimal postcomposition of the the linear operator M and the function f
P_C	:	Projection on the closed convex set C
prox_f	:	Proximal operator associated with the function f
Q_A	:	Resolvent of the operator A
$\mathrm{ri}(C)$	:	Relative interior of the set C
$\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$	:	Euclidean spaces
$\mathcal{Z}(A)$	:	Set of zeros of a set-valued operator A (P. 38)
∂f	:	Subdifferential of the function f
ι_C	:	Indicator function, $\iota_C(x) = 0$ if $x \in C$ and ∞ otherwise
$\Gamma_0(\mathcal{X})$	:	Set of convex, proper and l.s.c. functions on the Euclidean space $\mathcal{X}$
∇f	:	Gradient of the function f
$\|\cdot\|$	:	Euclidean norm of a vector or spectral norm of a matrix
$a \wedge b$	:	minimum of a and b
$a \vee b$	:	maximum of a and b

Bibliography

Bauschke, H. H. and Combettes, P. L. (2011). *Convex analysis and monotone operator theory in Hilbert spaces*. Springer.

Borwein, J. and Lewis, A. (2006). *Convex Analysis and Nonlinear Optimization: Theory and Examples*. CMS Books in Mathematics. Springer.

Boyd, S., Parikh, N., Chu, E., Peleato, B., and Eckstein, J. (2011). Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends® in Machine Learning*, 3(1):1–122. 44

Boyd, S. and Vandenberghe, L. (2009). *Convex optimization*. Cambridge university press.

Brézis, H. (1973). *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. North-Holland mathematics studies. Elsevier Science, Burlington, MA. 48

Brézis, H. (1987). Analyse fonctionnelle, 2e tirage.

Clarke, F. H., Ledyaev, Y. S., Stern, R. J., and Wolenski, P. R. (1998). *Nonsmooth analysis and control theory*, volume 178 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. 54, 61

Condat, L. (2013). A primal-dual splitting method for convex optimization involving Lipschitzian, proximable and linear composite terms. *J. Optim. Theory Appl.*, 158(2):460–479. 51

Hiriart-Urruty, J.-B. and Lemaréchal, C. (2012). *Fundamentals of convex analysis*. Springer Science & Business Media. 18

Nesterov, Y. (2004). *Introductory lectures on convex optimization: A basic course*, volume 87. Springer.

Rockafellar, R. T. (2015). *Convex analysis*. Princeton university press. 6, 59

Rockafellar, R. T. and Wets, R. J.-B. (1998). *Variational analysis*, volume 317. Springer. 13, 61