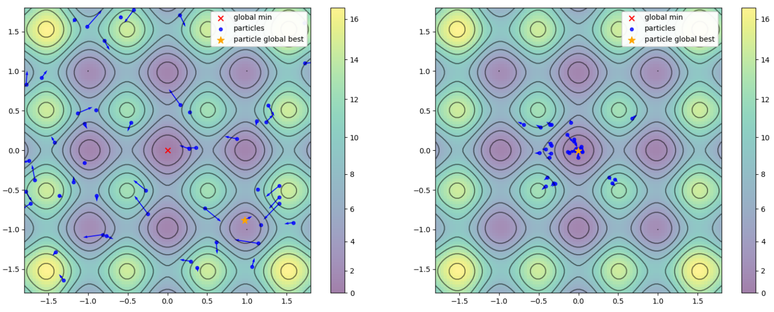
Research internship proposal: Theory of nonconvex optimization using particle swarm algorithms

Pascal Bianchi
pascal.bianchi@telecom-paris.fr

Radu Dragomir
dragomir@telecom-paris.fr



1 Project description

Particle swarm optimization (PSO) is a class of nature-inspired numerical algorithms for finding the global minimum of a nonconvex function F on some compact set of $\mathbb{R}^d$ [4, 2, 1]. PSO methods mimic the social behavior of a group of animals (such as flocks of birds or fish) that cooperate using collective intelligence. They are popular for solving medium-sized nonconvex optimization problems due to their simplicity –requiring only function evaluations rather than gradient computations– and their efficient exploration of the search space. PSO algorithms are used in a large variety of applications in data science, scientific research and engineering; see [7, 8].

PSO methods are essentially **heuristic**, meaning that few theoretical guarantees are available and that the parameters are usually chosen through empirical considerations. The goal of this research project is then to advance our theoretical understanding of PSO algorithms.

Description of PSO. PSO methods track the evolution of N particles $(X_1(t)\dots X_N(t))$ in $\mathbb{R}^d$ for T iterations $t = 0, 1 \dots T$. At each iteration, the particle speed vectors $(V_1(t)\dots V_N(t))$ are updated by combining three influences:

- a **local optimization** term drives the i-th particle towards its “personal best” position known at time t, formally defined as

$$\text{pbest}_i(t) = \text{argmin}\{F(X) \,:\, X \in \{X_i(0), \dots X_i(t-1)\}\},$$

- a **social influence** term that moves each particle towards the “global best” position known to **all particles**, formally defined as

$$\text{gbest}(t) = \text{argmin}\{F(X) \,:\, X \in \{\text{pbest}_1(t), \dots \text{pbest}_N(t)\}\}$$

- a **noise** term $\epsilon_i(t)$ that encourages exploration of the space by adding randomness.

The algorithms works by initializing the positions and speeds to random values, and performs the following updates for $t = 0, 1 \dots$ and $i = 1, \dots N$:

$$X_i(t+1) = X_i(t) + V_i(t) \quad (1)$$
$$V_i(t+1) = \lambda V_i(t) + \underbrace{\alpha_i(t)\big(\text{pbest}_i(t) - X_i(t)\big)}_{\text{local term}} + \underbrace{\beta_i(t)\big(\text{gbest}(t) - X_i(t)\big)}_{\text{social term}} + \underbrace{\epsilon_i(t)}_{\text{noise}}, \quad (2)$$

where $\lambda, \alpha_i(t), \beta_i(t)$ are user-defined coefficients. We then hope that the particles, or at least the global best position, converge towards the global minimizer of function F.

The goal of this project is to answer the following question:

When is the method guaranteed to converge to the global minimizer?

This would allow to indentify favorable classes of function on which the algorithm works well, as well as guidelines for chosing the method parameters.

Current approaches. Most of the litterature has focused on proving convergence of PSO to a *stable point* of the process [9] or to a *local minimizer* of the function [10].

Obtaining convergence guarantees to a global minimizer is much more difficult. Such result has only been obtained only in idealized situations, for simplified variants of PSO. In particular, **consensus-based optimization** (CBO) [6] is a variant obtained by removing the local and momentum terms from (2). Two approximations are then made:

- **continuous-time limit:** the discrete process (1)-(2) is approximated as a continuous-time stochastic differential equation (SDE);
- **mean-field limit:** for a large number N of particles, the behavior approaches that of a continuous measure μ on $\mathbb{R}^d$ which satisfies the partial differential equation (PDE)

$$\frac{\partial \mu(t)}{\partial t} = \operatorname{div}\left(\mu(t)\kappa[\mu(t)]\right) + \Delta\left(\sigma[\mu(t)]\mu(t)\right), \quad (3)$$

  for some functions κ, μ acting on measures. This is a Fokker-Planck-type equation describing a **diffusion process** [5].

In some favorable scenarios, we can then show that the solutions of (3) concentrate on the global minimizers of F as $t \to \infty$, proving the consistency of the method for this idealized context [6, 3].

Research Objective. Existing results are important blueprints for understanding why particle swarms work in practice. However, these studies have significant limitations. They consider an idealized version of PSO, where the number of particles is assumed to be infinite ($N = \infty$). Even under this assumption, these results only establish that the algorithm converges within *some neighborhood* of the global minimizer, falling short of demonstrating pointwise convergence.

The goal of this project is to:

- **establish the global convergence of PSO in realistic scenarios** as the number of particles N and the number of iterations become large,
- determine convergence rates,
- develop new PSO variants based on more complex interaction models (e.g., bird flocks).

2 Internship details

The internship will take place at LTCI/Télécom Paris, within the S2A team (Signal, Statistics, and Learning) and Hi! Paris. The team benefits from a large group of Ph.D. students and researchers actively collaborating in the fields of Optimization, Statistics, and Machine Learning.

The intern will start by doing a litterature review and exploring a research direction among the ones mentioned above, with the goal of pursuing with a PhD thesis on the topic.

The mathematical tools involve stochastic analysis, stochastic/partial differential equations and non-convex optimization theory. The candidate should have a strong background in mathematics, good coding skills, and appetite for theoretical research.

References

[1] Mohammad Reza Bonyadi and Zbigniew Michalewicz. Particle swarm optimization for single objective continuous space problems: A review. *Evolutionary Computation*, 25(1):1–54, 03 2017.

[2] Andries P. Engelbrecht. *Particle Swarm Optimization*, chapter 16, pages 289–358. Computational Intelligence: An Introduction. John Wiley & Sons, Ltd, 2007.

[3] Massimo Fornasier, Timo Klock, and Konstantin Riedl. Consensus-based optimization methods converge globally. *SIAM Journal on Optimization*, 34(3):2973–3004, September 2024.

[4] J. Kennedy and R. Eberhart. Particle swarm optimization. In *Proceedings of ICNN'95 - International Conference on Neural Networks*, volume 4, pages 1942–1948 vol.4, 1995.

[5] Grigorios A. Pavliotis. *Diffusion Processes*, pages 29–54. Springer New York, New York, NY, 2014.

[6] René Pinnau, Claudia Totzeck, Oliver Tse, and Stephan Martin. A consensus-based model for global optimization and its mean-field limit. *Mathematical Models and Methods in Applied Sciences*, 27(01):183–204, January 2017.

[7] Riccardo Poli. Analysis of the publications on the applications of particle swarm optimisation. *Journal of Artificial Evolution and Applications*, 2008(1), January 2008.

[8] Tareq M. Shami, Ayman A. El-Saleh, Mohammed Alswaitti, Qasem Al-Tashi, Mhd Amen Summakieh, and Seyedali Mirjalili. Particle swarm optimization: A comprehensive survey. *IEEE Access*, 10:10031–10061, 2022.

[9] Ioan Cristian Trelea. The particle swarm optimization algorithm: convergence analysis and parameter selection. *Information Processing Letters*, 85(6):317–325, 2003.

[10] Frans van den Bergh and Andries Petrus Engelbrecht. A convergence proof for the particle swarm optimiser. *Fundam. Inf.*, 105(4):341–374, December 2010. ISSN 0169-2968.